
Linear Combinations of Variables

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Applied Multivariate Statistics

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Outline

● Outline

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- Motivation
- Population
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- Example
- Two sets of Variables (and more linear algebra)

Reading: Johnson & Wichern pages 75–75, 149–192



Motivation

- A major tool for data reduction & insight into why reject H_o regarding μ_s .
- Definition: A **Linear Combination** of p (random) variables X_1, X_2, \dots, X_p is $a_1X_1 + a_2X_2 + \dots + a_pX_p$.

Let

$$\mathbf{a}' = (a_1, a_2, \dots, a_p) \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}$$

A linear combination in terms of vector operations is

$$\mathbf{a}' \mathbf{X}_{p \times 1}$$

- The linear combination $\mathbf{a}' \mathbf{X}$ is itself a random variable.

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Mean of Linear Combination

The mean of $a'X$ is

$$\begin{aligned} E(a'X) &= E(a_1X_1 + a_2X_2 + \cdots + a_pX_p) \\ &= E(a_1X_1) + E(a_2X_2) + \cdots + E(a_pX_p) \\ &= a_1E(X_1) + a_2E(X_2) + \cdots + a_pE(X_p) \\ &= a_1\mu_1 + a_2\mu_2 + \cdots + a_p\mu_p \\ &= a'\mu \end{aligned}$$

where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}$$

Know this:

$$E(a'X) = a'\mu$$

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Variance of a Linear Combination

The variance of $a'X$ is

$$\begin{aligned}\text{var}(a'X) &= \text{var}(a_1X_1 + a_2X_2 + \cdots + a_pX_p) \\ &= a_1^2\text{var}(X_1) + a_2^2\text{var}(X_2) + \cdots + a_p^2\text{var}(X_p) \\ &\quad + a_1a_2\text{cov}(X_1, X_2) + \cdots + a_{p-1}a_p\text{cov}(X_{p-1}, X_p) \\ &= \sum_{i=1}^p a_i^2\sigma_{ii} + \sum_{i \neq k} \sum_k a_i a_k \sigma_{ik} = \sum_i \sum_k a_i a_k \sigma_{ik}\end{aligned}$$

More compactly, for $p = 2$,

$$\begin{aligned}\text{var}(a'X) &= a_1^2\sigma_{11} + 2a_1a_2\sigma_{12} + a_2^2\sigma_{22} \\ &= (a_1, a_2) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= a'\Sigma a\end{aligned}$$

A Quadratic Form.

For any p , $\text{var}(a'X) = a'\Sigma a$

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Results if we add a constant

If we add a constant to the linear combination,

$$a_1X_1 + a_2X_2 + \cdots + a_pX_p + c = \mathbf{a}'\mathbf{X} + c$$

Then

Mean: Changes location

$$E(\mathbf{a}'\mathbf{X} + c) = \mathbf{a}'E(\mathbf{X}) + c = \mathbf{a}'\boldsymbol{\mu} + c$$

Variance: No change

$$\text{var}(\mathbf{a}'\mathbf{X} + c) = \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}$$

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Multiple Linear Combinations

Sometimes one just isn't enough...

Suppose that there are q linear combinations of the p random variables,

$$y_1 = a_{11}X_1 + a_{12}X_2 + \cdots + a_{1p}X_p$$

$$y_2 = a_{21}X_1 + a_{22}X_2 + \cdots + a_{2p}X_p$$

$$\vdots$$

$$y_q = a_{q1}X_1 + a_{q2}X_2 + \cdots + a_{qp}X_p$$

In matrix form,

$$\mathbf{y}_{q \times 1} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & \cdots & a_{qp} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} = \mathbf{A}_{q \times p} \mathbf{X}_{p \times 1}$$

No restriction on q , but usually $q \leq p$

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Mean and Covariance for q

$$\mathbf{y}_{q \times 1} = \mathbf{A}_{q \times p} \mathbf{X}_{p \times 1}$$

The Mean vector for \mathbf{y} ,

$$\boldsymbol{\mu}_y = E(\mathbf{y}) = E(\mathbf{A}\mathbf{X}) = \mathbf{A}E(\mathbf{X}) = \mathbf{A}\boldsymbol{\mu}_x$$

The Covariance matrix for \mathbf{y} ,

$$\boldsymbol{\Sigma}_y = \text{cov}(\mathbf{y}) = \text{cov}(\mathbf{A}\mathbf{X}) = \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$$

These are 2 very important results!

$$\boldsymbol{\mu}_y = \mathbf{A}\boldsymbol{\mu}_x$$

$$\boldsymbol{\Sigma}_y = \mathbf{A}\boldsymbol{\Sigma}_x\mathbf{A}'$$

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Kind of an Example

Let \mathbf{X} be a random vector

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad \text{with} \quad \boldsymbol{\mu}_x = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Suppose we're interested in

■ The average: $Y_1 = (1/3)(X_1 + X_2 + X_3)$

■ Contrast: $Y_2 = (X_1 - X_2)$

$$\boldsymbol{\mu}_y = \begin{pmatrix} \mu_{y_1} \\ \mu_{y_2} \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \end{pmatrix} E \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(\mu_1 + \mu_2 + \mu_3) \\ \mu_1 - \mu_2 \end{pmatrix}$$

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... and the covariance matrix of Y ,

$$\Sigma_y = A \Sigma_x A' = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1/3 & 1 \\ 1/3 & -1 \\ 1/3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3}(\sigma_{11} + \sigma_{21} + \sigma_{31}) & \frac{1}{3}(\sigma_{12} + \sigma_{22} + \sigma_{32}) & \frac{1}{3}(\sigma_{13} + \sigma_{23} + \sigma_{33}) \\ (\sigma_{11} - \sigma_{21}) & (\sigma_{12} - \sigma_{22}) & (\sigma_{13} - \sigma_{23}) \end{pmatrix} \begin{pmatrix} 1/3 & 1 \\ 1/3 & -1 \\ 1/3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{9}(\sigma_{11} + \sigma_{22} + \sigma_{33} + 2(\sigma_{12} + \sigma_{13} + \sigma_{23})) & \frac{1}{3}(\sigma_{11} - \sigma_{22} + \sigma_{13} - \sigma_{23}) \\ \frac{1}{3}(\sigma_{11} - \sigma_{22} + \sigma_{13} - \sigma_{23}) & (\sigma_{11} + \sigma_{22} - 2\sigma_{12}) \end{pmatrix}$$

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Estimation of μ_y

1st just one linear transformation (combination, composite, etc)

Let $\mathbf{a}' = (a_1, a_2, \dots, a_p)$ and $\mathbf{x}' = (x_1, x_2, \dots, x_p)$ and

$$y = \mathbf{a}'\mathbf{x} = \sum_{i=1}^p a_i x_i$$

If we have n cases, the sample observation of the linear transformation for the j^{th} case (individual) is y_j ,

$$\mathbf{a}'\mathbf{x}_j = a_1 x_{j1} + a_2 x_{j2} + \dots + a_p x_{jp} \quad \text{for } j = 1, \dots, n$$

$$\begin{aligned} \text{Sample Mean of } y : \quad \bar{y} &= \frac{1}{n}(y_1 + y_2 + \dots + y_n) \\ &= \frac{1}{n}(\mathbf{a}'\mathbf{x}_1 + \mathbf{a}'\mathbf{x}_2 + \dots + \mathbf{a}'\mathbf{x}_n) \\ &= \mathbf{a}' \frac{1}{n} \sum_{j=1}^n \mathbf{x}_j \\ &= \mathbf{a}' \bar{\mathbf{x}} \end{aligned}$$

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● Estimation of μ_y

● Estimation of $\text{var}(y)$

● Estimation of \bar{y} and $\text{var}(y)$

● Sample Covariance

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Estimation of $\text{var}(y)$

Sample variance of y :

$$\begin{aligned}\text{var}(y) &= \frac{1}{n-1} \sum_{j=1}^n (y_j - \bar{y})^2 \\ &= \frac{1}{n-1} \sum_{j=1}^n (\mathbf{a}'\mathbf{x}_j - \mathbf{a}'\bar{\mathbf{x}})^2 \\ &= \frac{1}{n-1} \sum_{j=1}^n (\mathbf{a}'\mathbf{x}_j - \mathbf{a}'\bar{\mathbf{x}}) \underbrace{(\mathbf{a}'\mathbf{x}_j - \mathbf{a}'\bar{\mathbf{x}})}_{\text{note: } \mathbf{a}'(\mathbf{x}_j - \bar{\mathbf{x}}) = (\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{a}} \\ &= \frac{1}{n-1} \sum_{j=1}^n \mathbf{a}'(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{a} \\ &= \mathbf{a}' \left(\frac{1}{n-1} \sum_{j=1}^n (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})' \right) \mathbf{a} \\ &= \mathbf{a}'\mathbf{S}\mathbf{a}\end{aligned}$$

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Estimation of \bar{y} and $\text{var}(y)$

$a'\bar{x}$ is the sample analog to $a'\mu$

$a'Sa$ is the sample analog to $a'\Sigma a$

Let's add a 2^{nd} linear combination of the vector \mathbf{X} :

$$Z = b_1X_1 + b_2X_2 + \cdots + b_pX_p = \mathbf{b}'\mathbf{X} = \sum_{i=1}^p b_iX_i$$

The j^{th} sample observation on variable Z is

$$z_j = \mathbf{b}'\mathbf{x}_j = b_1x_{j1} + b_2x_{j2} + \cdots + b_px_{jp}$$

Applying above results gives us

$$\text{Sample mean: } \bar{z} = \mathbf{b}'\bar{\mathbf{x}}$$

$$\text{Sample variance: } s_z^2 = \mathbf{b}'S\mathbf{b}$$

What's the covariance between y and z ?

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Sample Covariance

Between y and z

$$\begin{aligned}\text{cov}(y, z) &= \frac{1}{n-1} \sum_{j=1}^n (y_j - \bar{y})(z_j - \bar{z}) \\ &= \frac{1}{n-1} \sum_{j=1}^n (\mathbf{a}'\mathbf{x}_j - \mathbf{a}'\bar{\mathbf{x}})(\mathbf{b}'\mathbf{x}_j - \mathbf{b}'\bar{\mathbf{x}}) \\ &= \frac{1}{n-1} \mathbf{a}' \left(\sum_{j=1}^n (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})' \right) \mathbf{b} \\ &= \mathbf{a}' \left(\frac{1}{n-1} \sum_{j=1}^n (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})' \right) \mathbf{b} \\ &= \mathbf{a}' \mathbf{S} \mathbf{b}\end{aligned}$$

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Example with Numbers

Suppose we have $n = 3$ and $p = 3$, and the data matrix is

$$\mathbf{X} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ 1 & 4 & 3 \end{pmatrix}$$

And we want to find the two linear combinations

$$\mathbf{b}'\mathbf{x} = b_1x_1 + b_2x_2 + b_3x_3$$

where $\mathbf{b}' = (1, 1, 1)$ and \mathbf{x} is a row of \mathbf{X} written as a column vector, and

$$\mathbf{c}'\mathbf{x} = c_1x_1 + c_2x_2 + c_3x_3$$

where $\mathbf{c} = (-2, 1, 1)$.

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● Note regarding $\text{var}(y)$'s and $\text{cov}(y_1, y_2)$

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Observations on “new” variables

Observations on these two “new” variables equal

$$b'X' = (1, 1, 1) \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 3 \end{pmatrix} = (7, 7, 8)$$

$$c'X' = (-2, 1, 1) \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 3 \end{pmatrix} = (4, 4, 5)$$

In terms of single matrix formula,

$$AX' = \begin{pmatrix} b' \\ c' \end{pmatrix} X' = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 7 & 8 \\ 4 & 4 & 5 \end{pmatrix}$$

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Mean and Covariance for x 's

The means on the original variables are

$$\bar{x}_1 = 1, \quad \bar{x}_2 = 3, \quad \text{and} \quad \bar{x}_3 = 3.333$$

Sample means of new variables computed using the scalar formula with the observations on the new variables (on the left) and matrix formula using the observations on the old variables (on the right):

$$\begin{pmatrix} \frac{1}{3}(7 + 7 + 8) \\ \frac{1}{3}(4 + 4 + 5) \end{pmatrix} = \begin{pmatrix} 7.33 \\ 4.33 \end{pmatrix} = \mathbf{A}\bar{\mathbf{x}} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3.33 \end{pmatrix} = \begin{pmatrix} 7.33 \\ 4.33 \end{pmatrix}$$

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Sample Variances & Covariance for x 's

Sample variances of the old variables:

$$s_{11} = 0$$

$$s_{22} = \frac{1}{2} [(2 - 3)^2 + (3 - 3)^2 + (4 - 3)^2] = 1$$

$$s_{33} = \frac{1}{2} [(4 - 3.33)^2 + (3 - 3.33)^2 + (3 - 3.33)^2] = \frac{1}{3}$$

Sample covariances between the old variables:

$$s_{12} = \frac{1}{2} [0(2 - 3) + 0(3 - 3) + 0(4 - 3)] = 0$$

$$s_{13} = \frac{1}{2} [0(4 - 3.33) + 0(3 - 3.33) + 0(3 - 3.33)] = 0$$

$$\begin{aligned} s_{23} &= \frac{1}{2} [(2 - 3)(4 - 3.33) + (3 - 3)(3 - 3.33) + (4 - 3)(3 - 3.33)] \\ &= \frac{1}{2} (-.67 + 0 - .33) = -\frac{1}{2} \end{aligned}$$

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Sample Covariance Matrices

The sample variance-covariance matrix of the x 's variables is

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

For the new variables (the y 's):

$$\begin{aligned} ASA' &= \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{2} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{aligned}$$

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Note regarding $\text{var}(y)$'s and $\text{cov}(y_i, y_2)$

The results on the previous slide are the same if we had used scalar formulas with the values of the y_{j1} and y_{j2} :

$$s_{11} = \frac{1}{2}[(7 - 7.33)^2 + (7 - 7.33)^2 + (8 - 7.33)^2] = \frac{1}{3}$$

$$s_{22} = \frac{1}{2}[(4 - 4.33)^2 + (4 - 4.33)^2 + (5 - 4.33)^2] = \frac{1}{3}$$

$$s_{12} = \frac{1}{2}[(7 - 7.33)(4 - 4.33) + (7 - 7.33)(4 - 4.33) + (8 - 7.33)(5 - 4.33)] = \frac{1}{3}$$

It was just by coincidence that all the values are $1/3$... bad choice of “data” on my part.

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Two Sets of Linear Combinations

If we have one set of q linear combinations and another set of r linear combinations of random variables \mathbf{X} , then

$$\mathbf{Y}_{q \times 1} = \mathbf{A}_{q \times p} \mathbf{X}_{p \times 1}$$

$$\mathbf{Z}_{r \times 1} = \mathbf{B}_{r \times p} \mathbf{X}_{p \times 1}$$

$$E(\mathbf{Y}) = \mathbf{A} \mu_x$$

$$E(\mathbf{Z}) = \mathbf{B} \mu_x$$

$$\Sigma_y = \mathbf{A} \Sigma_x \mathbf{A}'$$

$$\Sigma_z = \mathbf{B} \Sigma_x \mathbf{B}'$$

$$\Sigma_{yz} = \mathbf{A} \Sigma_x \mathbf{B}'$$

This is just an application of what we already did. . .

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● Two Sets of Linear Combinations

● More Linear Algebra:

Partitioned Matrices

● Partitioned mean and covariance matrices

● Operations on Partitioned Vectors & Matrices



More Linear Algebra: Partitioned Matrices

Let matrix $A_{k \times p}$ be **Partitioned** into two parts as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{q1} & a_{q2} & \cdots & a_{qp} \\ a_{q+1,1} & a_{q+1,2} & \cdots & a_{q+1,p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{q+r,1} & a_{q+r,2} & \cdots & a_{q+r,p} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{1,(q \times p)} \\ \mathbf{A}_{2,(r \times p)} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{q \times p} \\ \mathbf{B}_{r \times p} \end{pmatrix}$$

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_q \\ Y_{q+1} \\ \vdots \\ Y_{q+r} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_{1,(q \times 1)} \\ \mathbf{Y}_{2,(r \times 1)} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_{(q \times 1)} \\ \mathbf{Z}_{(r \times 1)} \end{pmatrix}$$

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Partitioned mean and covariance matrices

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$$\boldsymbol{\mu}_{(q+r) \times 1} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \mu_{q+1,1} \\ \vdots \\ \mu_{q+r} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{1,(q \times 1)} \\ \boldsymbol{\mu}_{2,(r \times 1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{Y,(q \times 1)} \\ \boldsymbol{\mu}_{Z,(r \times 1)} \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{(q+r),(q+r)} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1,q+r} \\ \vdots & \ddots & \cdots & \vdots \\ \sigma_{q1} & \sigma_{q2} & \cdots & \sigma_{q,q+r} \\ \sigma_{q+1,1} & \sigma_{q+1,2} & \cdots & \sigma_{q+1,q+r} \\ \vdots & \ddots & \cdots & \vdots \\ \sigma_{q+r,1} & \sigma_{q+r,2} & \cdots & \sigma_{q+r,q+r} \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_{yy,(q \times q)} & \boldsymbol{\Sigma}_{yz,(q \times r)} \\ \hline \boldsymbol{\Sigma}_{zy,(r \times q)} & \boldsymbol{\Sigma}_{zz,(r \times r)} \end{array} \right)$$

where $\boldsymbol{\Sigma}_{yz} = \boldsymbol{\Sigma}'_{zy}$.



Operations on Partitioned Vectors & Matrices

Same; that is, treat each sub-matrix (vector) as an element of a matrix (vector).

For example

$$\begin{pmatrix} \mathbf{A}_{q \times p} \\ \mathbf{B}_{r \times p} \end{pmatrix} \mathbf{X}_{p \times 1} = \begin{pmatrix} (\mathbf{A}\mathbf{X})_{q \times 1} \\ (\mathbf{B}\mathbf{X})_{r \times 1} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_{q \times 1} \\ \mathbf{Z}_{r \times 1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}_{q \times p} \\ \mathbf{B}_{r \times p} \end{pmatrix} \boldsymbol{\mu}_{\mathbf{x}(p \times 1)} = \begin{pmatrix} (\mathbf{A}\boldsymbol{\mu})_{q \times 1} \\ (\mathbf{B}\boldsymbol{\mu})_{r \times 1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{\mathbf{y},(q \times 1)} \\ \boldsymbol{\mu}_{\mathbf{z},(r \times 1)} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}_{q \times p} \\ \mathbf{B}_{r \times p} \end{pmatrix} \boldsymbol{\Sigma}_{p \times p} \left(\mathbf{A}'_{p \times q} \mid \mathbf{B}'_{p \times q} \right) = \begin{pmatrix} \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' & \mathbf{A}\boldsymbol{\Sigma}\mathbf{B}' \\ \mathbf{B}\boldsymbol{\Sigma}\mathbf{A}' & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{yy} & \boldsymbol{\Sigma}_{yz} \\ \boldsymbol{\Sigma}_{zy} & \boldsymbol{\Sigma}_{zz} \end{pmatrix}$$

which gives us the results for linear combinations for 2 sets of variables.

● Outline

Motivation

Population

Estimation

Example

Two Sets

● Two Sets of Linear

Combinations

● More Linear Algebra:

Partitioned Matrices

● Partitioned mean and covariance matrices

● Operations on Partitioned Vectors & Matrices