Logit Models for Multicategory Responses

Situation:

- One response variable Y with J levels.
- One or more explanatory or predictor variables. The predictor variables may be quantitative, qualitative or both.

Model: Logistic regression.

What if you have multiple response variables?

There are 3 basic ways in which logistic regression for multicategory responses is different from logistic regression for dichotomous or binary data.

1. Forming Logits.
2. The sampling model.
3. Connections with other models such as Poisson regression and loglinear models.

Additional (general) References:

Agresti, A. (1990). *Categorical Data Analysis*. NY: Wiley.

Long, J.S. (1997). *Regression Models for Categorical and Limited Dependent Variables*. Thousand Oaks, CA: Sage.

Powers, D.A. & Xie, Y. (2000). *Statistical Methods for Categorical Data Analysis*. San Diego, CA: Academic Press.

Additional References on Fitting (Conditional) Multinomial Models using SAS:

SAS Institute (1995). *Logistic Regression Examples Using the SAS System,* (version 6). Cary, NC: SAS Institute.

Kuhfeld, W.F. (2001). *Marketing Research Methods in the SAS System,* Version 8.2 Edition, TS-650. Cary, NC: SAS Institute. http://www.sas.com/service/techsup/tnote/tnote_stat.html (reports TS-650A – TS-560I).

Search http://www.sas.com/service/techsup/tnote/tnote_stat.html.

Forming Logits

When $J = 2$, Y is dichotomous and we can model logs of odds that an event occurs or does not occur. There is only 1 logit that we can form

$$\text{logit}(\pi) = \log\left(\frac{\pi}{1-\pi}\right)$$

When $J > 2$, ...

- We have a multicategory or "polytomous" or "polychotomous" response variable.
- There are $J(J-1)/2$ logits (odds) that we can form, but only $(J-1)$ are non-redundant.
- There are different ways to form a set of $(J-1)$ non-redundant logits.

How to "dichotomized" the response Y?

1. Nomnial Y —
   (a) "Baseline" logit models or "Multinomial" logistic regression.
   (b) "Conditional" or "Multinomial" logit models.
2. Ordinal Y —
   (a) Cumulative logits.
   (b) Adjacent categories.
   (c) Continuation ratios.

(These are the most common and generally the most useful ones).

Sampling Model.

With dichotomous Y, at each combination of levels of the explanatory variables, we assume data arise from a Binomial distribution.

e.g., "Brewed, bottled or powered: Testing the health effects of tea." (Nov, 1999). *Consumer Reports*, pp. 60–61.

Results of clinical study by Junshi Chen of Chinese Academy.

	Pre-cancerous Mouth Lesions		
	Worse or no change	Shrink (improve)	
green tea	18	11	29
placebo	27	3	30
			59

With $J > 2$, at each combination of levels of the explanatory variables, we assume a **Multinomial distribution**.
e.g.,

	Pre-cancerous Mouth Lesions			
	Worse	No change	Shrink	
green tea				29
placebo				30

Connections with other Models.

1. Some are equivalent to Poisson regression or loglinear models.
2. Some can be derived from (equivalent to) discrete choice models (e.g., Luce, McFadden).
3. Those that are equivalent to conditional multinomial models are equivalent to proportional hazard models (models for survival data), which is equivalent to Poisson regression model.
4. Some can be derived from latent variable models.
5. Some multicategory logit models are very similar to IRT models in terms of their parametric form. The difference between them is that in the IRT models, the predictor is unobserved (latent), and in the model we discuss here, the predictor variable is observed.
6. Others.

Nominal Response

Y has J categories (order is irrelevant).

$\{\pi_1, \pi_2, \ldots, \pi_J\}$ are probabilities that response is in each category.

$\pi_1 + \pi_2 + \ldots + \pi_J = \Sigma_{j=1}^{J} \pi_j = 1.$

The probability distribution for the number of outcomes that occur in the J categories for a sample of n independent observations is **Multinomial**.

- The Binomial distribution is a special case of the Multinomial.
- The multinomial distribution depends on n & $\{\pi_1, \pi_2, \ldots, \pi_J\}$.
- The multinomial distribution gives the probability for each way to classify the n observations into the J categories of the response variable.

  For example, the possible ways to classify $n = 2$ observations into $J = 3$ categories is

y_1	y_2	y_3
2	0	0
1	1	0
1	0	1
0	2	0
0	1	1
0	0	2

Multicategory Logit Models for Nominal Responses.

Possibilities:

1. Baseline or Multinomial logistic regression model. Use characteristics of individuals as predictor variables.

   The parameters differ for each category of the response variable.

2. Conditional Logit model. Use characteristics of the categories of the response variable as the predictors.

   The model parameters are the same for each category of the response variable.

3. Conditional or Mixed logit model. Uses characteristics or attributes of the individuals <u>and</u> the categories as predictor variables.

There is not a standard terminology for these models.

- Agresti (90) regarding 2: "Originally referred to by McFadden as a *conditional logit* model, it is now usually called the *multinomial logit* model."
- Long (97): calls 1 "multinomial logit" model and calls 2 "conditional logit" model.
- Powers & Xie (00) regarding 2 & 3, "However, it is often called a multinominal logit model, leading to a great deal of confusion."

Baseline/Multinomial Category Logit Model

The models we consider here give a simultaneous representation (summary, description) of the odds of being in one category relative to being in another category for all pairs of categories.

We need a set of $(J-1)$ non-redundant odds (logits). Given this, we can figure out the odds for any pair of categories.

This model is basically just an extension of the binary logistic regression model.

Consider the HSB data:

Response variable is High school program (HSP) type where

1. General
2. Academic
3. Vo/Tech

Explanatory variables maybe

- Mean of the five achievement test scores, which is numerical/continuous (x_i).
- Socio-economic status, which will be either nominal (β_i^s) or ordinal/numerical (s_i).
- School type, which would be nominal (public, private).

We could fit a binary logit model to each pair of program types:

$$\begin{aligned}
\log\left(\frac{\text{general}}{\text{academic}}\right) = \log\left(\frac{\pi_1(x_i)}{\pi_2(x_i)}\right) &= \alpha_1 + \beta_1 x_i \\
\log\left(\frac{\text{academic}}{\text{vo/tech}}\right) = \log\left(\frac{\pi_2(x_i)}{\pi_3(x_i)}\right) &= \alpha_2 + \beta_2 x_i \\
\log\left(\frac{\text{general}}{\text{vo/tech}}\right) = \log\left(\frac{\pi_1(x_i)}{\pi_3(x_i)}\right) &= \alpha_3 + \beta_3 x_i
\end{aligned}$$

We can write one of the odds in terms of the other 2,

$$\left(\frac{\pi_1(x_i)}{\pi_2(x_i)}\right)\left(\frac{\pi_2(x_i)}{\pi_3(x_i)}\right) = \frac{\pi_1(x_i)}{\pi_3(x_i)},$$

Therefore,we can find the model parameters of one from the other 2,

$$\begin{aligned}
\log\left(\frac{\pi_1(x_i)}{\pi_2(x_i)}\right) + \log\left(\frac{\pi_2(x_i)}{\pi_3(x_i)}\right) &= \log\left(\frac{\pi_1(x_i)}{\pi_3(x_i)}\right) \\
(\alpha_1 + \beta_1 x_i) + (\alpha_2 + \beta_2 x_i) &= \alpha_3 + \beta_3 x_i
\end{aligned}$$

which means that in the *Population*

$$\alpha_1 + \alpha_2 = \alpha_3$$

$$\beta_1 + \beta_2 = \beta_3$$

With sample data,

- The estimates from separate binary logit models are *consistent* estimators of the parameters of the model.
- Estimates from fitting separate binary logit models will not yield the equality between the parameters that holds in the population.

$$\hat{\alpha}_1 + \hat{\alpha}_2 \neq \hat{\alpha}_3$$
$$\hat{\beta}_1 + \hat{\beta}_2 \neq \hat{\beta}_3$$

Solution: simultaneous estimation

- Enforces the logical relationships among parameters.
- Uses the data more *efficiently*, which means that the standard errors of parameter estimates are smaller with simultaneous estimation.

Problem: A difficulty is that there are a large number of comparisons and some of them are redundant.

Solution: Choose one of the categories and treat it as a "baseline." Depending on the study and response variable,

- There maybe a natural choice for the baseline category.
- The choice maybe arbitrary.

Baseline Category Logit Model

For convenience, we'll use the last level of the response variable as the baseline (i.e., the Jth level or category).

$$\log\left(\frac{\pi_{ij}}{\pi_{iJ}}\right) \qquad \text{for} \quad j = 1, \ldots, J-1$$

The baseline category logit model with one explanatory variable x is

$$\log\left(\frac{\pi_{ij}}{\pi_{iJ}}\right) = \alpha_j + \beta_j x_i \qquad \text{for} \quad j = 1, \ldots, J-1$$

- For $J = 2$, this is just regular (binary) logistic regression.

$$\text{logit}(\pi) = \alpha + \beta x$$

- For $J > 2$, α and β can differ depending on which 2 categories are being compared.
- The odds for any pair of categories of Y that can be formed are a function of the parameters of the model.

Example: the HSB data where

Response variable is High school program (HSP) type where

1. General
2. Academic
3. Vo/Tech

Explanatory variable is the mean of the five achievement test scores, which is numerical/continuous (x_i).

For this example, we have $(3-1) = 2$ non-redundant logits (odds):

$$\log\left(\frac{\text{general}}{\text{vo/tech}}\right) = \log\left(\frac{\pi_1}{\pi_3}\right) = \alpha_1 + \beta_1 x$$

$$\log\left(\frac{\text{academic}}{\text{vo/tech}}\right) = \log\left(\frac{\pi_2}{\pi_3}\right) = \alpha_2 + \beta_2 x$$

The logit for (1) general and (2) academic equals

$$\begin{aligned}
\log\left(\frac{\pi_1}{\pi_2}\right) &= \log\left(\frac{\pi_1/\pi_3}{\pi_2/\pi_3}\right) \\
&= \log(\pi_1/\pi_3) - \log(\pi_2/\pi_3) \\
&= (\alpha_1 + \beta_1 x) - (\alpha_2 + \beta_2 x) \\
&= (\alpha_1 - \alpha_2) + (\beta_1 - \beta_2)x
\end{aligned}$$

The differences $(\beta_1 - \beta_2)$ are known as "contrasts".

Caution: You must be certain what the computer program that you use to estimate the model is doing.

- Programs that explicitly estimate the "baseline" logit model generally either set $\beta_1 = 0$ or set $\beta_J = 0$, and some set the sum $\Sigma_j \beta_j = 0$.
- Programs that fit the "multinomial" logit model may set $\beta_1 = 0$, $\beta_J = 0$, or $\Sigma_j \beta_j = 0$.

Again. . . the estimation should be simultaneous, because

- Simultaneous fitting is more efficient.
- The standard errors of parameter estimates are smaller when model is fit all at once.
- Want to impose the logical relationships among the parameters.

In SAS, this model can be estimated using either

- CATMOD
- GENMOD

Example: estimated model for High School and Beyond

$$\begin{aligned} \text{general/votech:} \quad \hat{\log}(\pi_1/\pi_3) &= -2.8996 + .0599x \\ \text{academic/votech:} \quad \hat{\log}(\pi_2/\pi_3) &= -7.9388 + .1699x \end{aligned}$$

and for comparing general and academic

$$\begin{aligned} \hat{\log}(\pi_1/\pi_2) &= \hat{\log}(\pi_1/\pi_3) - \hat{\log}(\pi_2/\pi_3) \\ &= -2.8996 + .0599x - (-7.9388 + .1699x) \\ &= 5.039 - .110x \end{aligned}$$

If we use either general or academic instead of votech as the baseline category, we get the exact same results.

The results using votech as the baseline were obtained using SAS/CATMOD:

```
data hsb;
 set sasdata.hsb;
 achieve=(RDG+WRTG+MATH+SCI+CIV)/5;

proc catmod;
 response logits;
 direct achieve;
 model hsp = achieve ;
 title 'Baseline/multinomial logit model: achieve';
```

Edited output from CATMOD:

```
The CATMOD Procedure

                  Data Summary

Response            HSP     Response Levels    3
Weight Variable     None    Populations      490
Data Set            HSB     Total Frequency  600
Frequency Missing   0       Observations     600
```

Population Profiles

Sample	achieve	Sample Size
1	32.94	1
2	52.76	1
3	33.74	1
4	52.78	1
5	34.98	1
6	52.8	2
7	35.3	1
8	52.82	1
9	35.36	1
.	.	.
.	.	.
.	.	.
488	52.64	1
489	52.66	1
490	52.7	1

Response Profiles

Response	HSP	
1	1	(General)
2	2	(Academic)
3	3	(Vo/Tech)

Maximum Likelihood Analysis

Iteration	Sub Iteration	-2 Log Likelihood	Convergence Criterion
0	0	1318.3347	1.0000
1	0	1087.9513	0.1748
2	0	1083.8083	0.003808
3	0	1083.7834	0.0000230
4	0	1083.7834	1.2814E-9

	Parameter Estimates			
Iteration	1	2	3	4
0	0	0	0	0
1	-1.8247	-6.7704	0.0349	0.1457
2	-2.8551	-7.8374	0.0590	0.1677
3	-2.8992	-7.9380	0.0599	0.1698
4	-2.8996	-7.9388	0.0599	0.1699

Maximum likelihood computations converged.

Maximum Likelihood Analysis of Variance

Source	DF	Chi-Square	Pr > ChiSq
Intercept	2	92.51	<.0001
achieve	2	112.71	<.0001
Likelihood Ratio	976	920.51	0.8971

Analysis of Maximum Likelihood Estimates

Parameter	Function Number	Estimate	Standard Error	Chi-Square	Pr > ChiSq
Intercept	1	-2.8996	0.8156	12.64	0.0004
	2	-7.9388	0.8438	88.51	<.0001
achieve	1	0.0599	0.0168	12.77	0.0004
	2	0.1699	0.0168	102.72	<.0001

Parameter		Estimate	e^{β}	ASE	Wald	p
Intercept	(general)	-2.8996		.8156	12.62	$< .001$
	(academic)	-7.9385		.8438	88.51	$< .001$
Achieve	(general)	.0599	1.06	.0169	12.77	$< .001$
	(academic)	.1699	1.19	.0168	102.72	$< .001$

Parameter		Estimate	e^{β}	ASE	Wald	p
Intercept	(general)	-2.8996		.8156	12.62	$< .001$
	(academic)	-7.9385		.8438	88.51	$< .001$
Achieve	(general)	.0599	1.06	.0169	12.77	$< .001$
	(academic)	.1699	1.19	.0168	102.72	$< .001$

Notes regarding interpretation:

- For comparing General to Academic,
  $\exp(\hat{\beta}_1 - \hat{\beta}_2) = \exp(.0599 - .1699) = \exp(-.110) = 1.12.$
- For a 10 point change in achievement, yields odds ratios

  General to Votech $= \exp(10(.0599)) = 1.82.$

  Academic to Votech $= \exp(10(.1699)) = 5.47.$

  General to Academic $= \exp(10(-.110)) = .33.$
  (or Academic to General $= 1/.33 = 3.00.$)

Trick to use SAS/GENMOD: re-arrange the data.

Consider the data as a 2–way, (Student × Program type) table:

		Program Type			
		general	academic	vo/tech	
	1	1	0	0	1
	2	1	0	0	1
Student	3	0	1	0	1
	⋮	⋮	⋮	⋮	⋮
	600	0	0	1	1

The saturated loglinear model for this table is

$$\log(\mu_{ij}) = \lambda + \lambda_i^S + \lambda_j^P + \lambda_{ij}^{SP}$$

Associated with each row/student is a numerical variable, "achieve". Consider "Student" as being ordinal and fit a nominal by ordinal loglinear model where the achieve test scores x_i are the category scores:

$$\log(\mu_{ij}) = \lambda + \lambda_i^S + \lambda_j^P + \beta_j^* x_i$$

We can convert the nominal by ordinal loglinear model into a logit model. For example, comparing General (1) and Vo/Tech (3):

$$\begin{aligned} \log\left(\frac{\mu_{i1}}{\mu_{i3}}\right) &= \log(\mu_{i1}) - \log(\mu_{i3}) \\ &= (\lambda_1^P - \lambda_3^P) + (\beta_1^* - \beta_3^*)x_i \\ &= \alpha_1 + \beta_1 x_i \end{aligned}$$

...our baseline/multinomial model.

Using SAS/GENMOD...

```
data hsp2;
input student hsp count achieve;
datalines;
   1  1  1  41.32
   1  2  0  41.32
   1  3  0  41.32
   ⋮  ⋮  ⋮      ⋮
 600  1  0  43.44
 600  2  0  43.44
 600  3  1  43.44

proc genmod;
class student hsp;
model count = student hsp hsp*achieve / link=log
dist=Poi;
```

"`Student`" ensures that the sum of each row of the fitted values equals 1 (fixed by design) — the λ_i^S's or "nuisance" parameters.

"`HSP`" ensures that the program type margin is fit perfectly — the λ_j^P's which gives us the α_j's in the logit model.

"`HSP*achieve`" — the β_j^* which gives the parameter estimates for the β_j's in the logit model.

Given that SAS/GENMOD sets $\lambda_3^P = 0$ and $\beta_3^* = 0$, you get the correct ASE errors for the α_j's and β_j's:

Since

$$\alpha_j = (\lambda_j^P - \lambda_3^P) = \lambda_j^P$$

The ASE of α_j simply equals the ASE of λ_j^P.

Since

$$\beta_j = (\beta_j^* - \beta_3^*) = \beta_j^*$$

The ASE of β_j simply equals ASE of β_j^*

Using either CATMOD or GENMOD, you can easily add more explanatory variables. For example,

<u>GENMOD</u>:

- SES as a nominal variable:

```
proc genmod;
class student hsp ses;
model count = student hsp hsp*achieve hsp*ses
  / link=log dist=Poi;
```

- SES as a numerical variable (e.g., SES=1,2,3)

```
proc genmod;
class student hsp;
model count = student hsp hsp*achieve hsp*ses
  / link=log dist=Poi;
```

CATMOD:

- SAS as a nominal variable:

```
proc catmod;
 response logits;
 direct achieve;
 model hsp = achieve ses ;
 title 'Achieve numerical and SES qualitative';
```

- SAS as a numerical (ordinal) variable:

```
proc catmod;
 response logits;
 direct achieve ses ;
 model hsp = achieve ses;
 title 'Both achieve and ses as numerical variable';
```

Other programs that I've used to fit multinomial models.

- Vermunt, J.K. (1997). *ℓ_{EM}: A General Program for the Analysis of Categorical Data*. Tilburg, University. www.kub.nl/fsw/organisatie/departmenten/mto/software2.html.
- *FORTRAN* program that I wrote.

Other programs that I know of (but haven't used).

- SYSTAT
- SPSS

To illustrate the need for simultaneous estimation ... the binary logistic regression model was fit separately to 2 of the 3 possible logits,

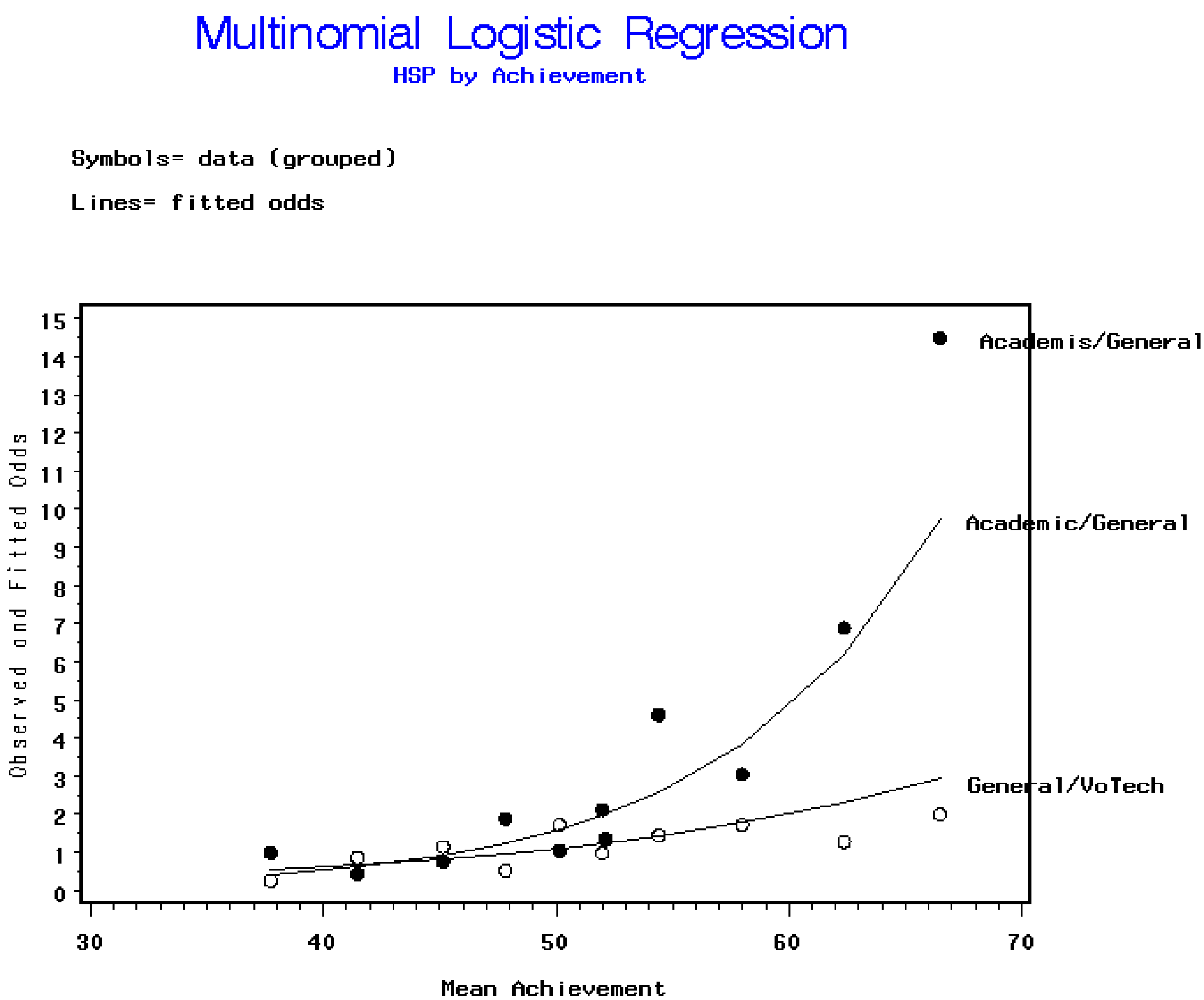
$$\log\left(\frac{\pi_1}{\pi_3}\right) = \alpha_1 + \beta_1 x$$

$$\log\left(\frac{\pi_2}{\pi_3}\right) = \alpha_2 + \beta_2 x$$

yields

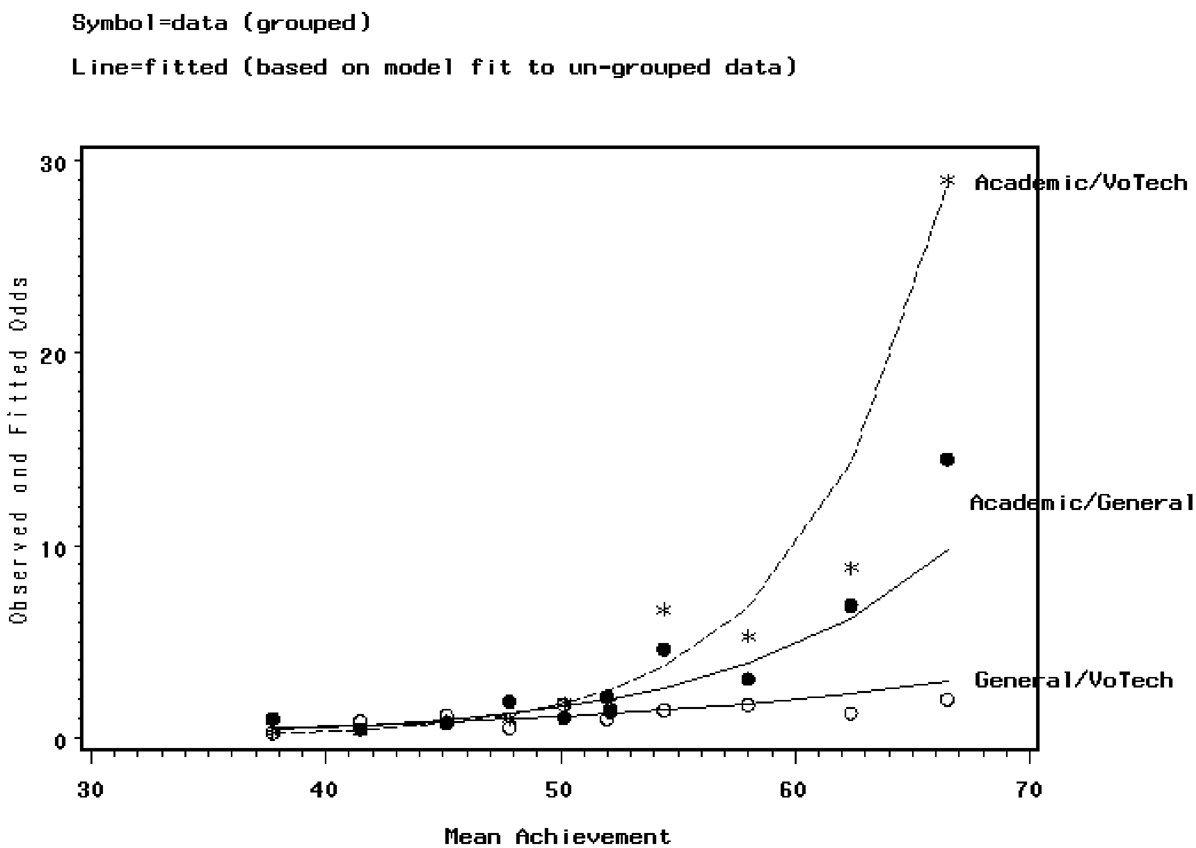
		Simultaneous Fit		Separate Fit	
Parameter		Estimate	ASE	Esiatmate	ASE
Intercept	(general)	-2.8996	.8156	-2.9656	.8342
	(academic)	-7.9385	.8438	-7.5311	.8572
Achieve	(general)	.0599	.0169	.0613	.0172
	(academic)	.1699	.0168	.1618	.0170

How well did the (simultaneous) model fit?



Multinomial Logistic Regression

HSP by Achievement



Computing Probabilities from the Baseline Logit Model.

Just as in logistic regression for $J = 2$, we can talk about (and interpret) baseline category logit model in terms of probabilities.

The probability of a response being in category j is

$$\pi_j = \frac{\exp(\alpha_j + \beta_j x)}{\Sigma_{h=1}^{J} \exp(\alpha_h + \beta_h x)}$$

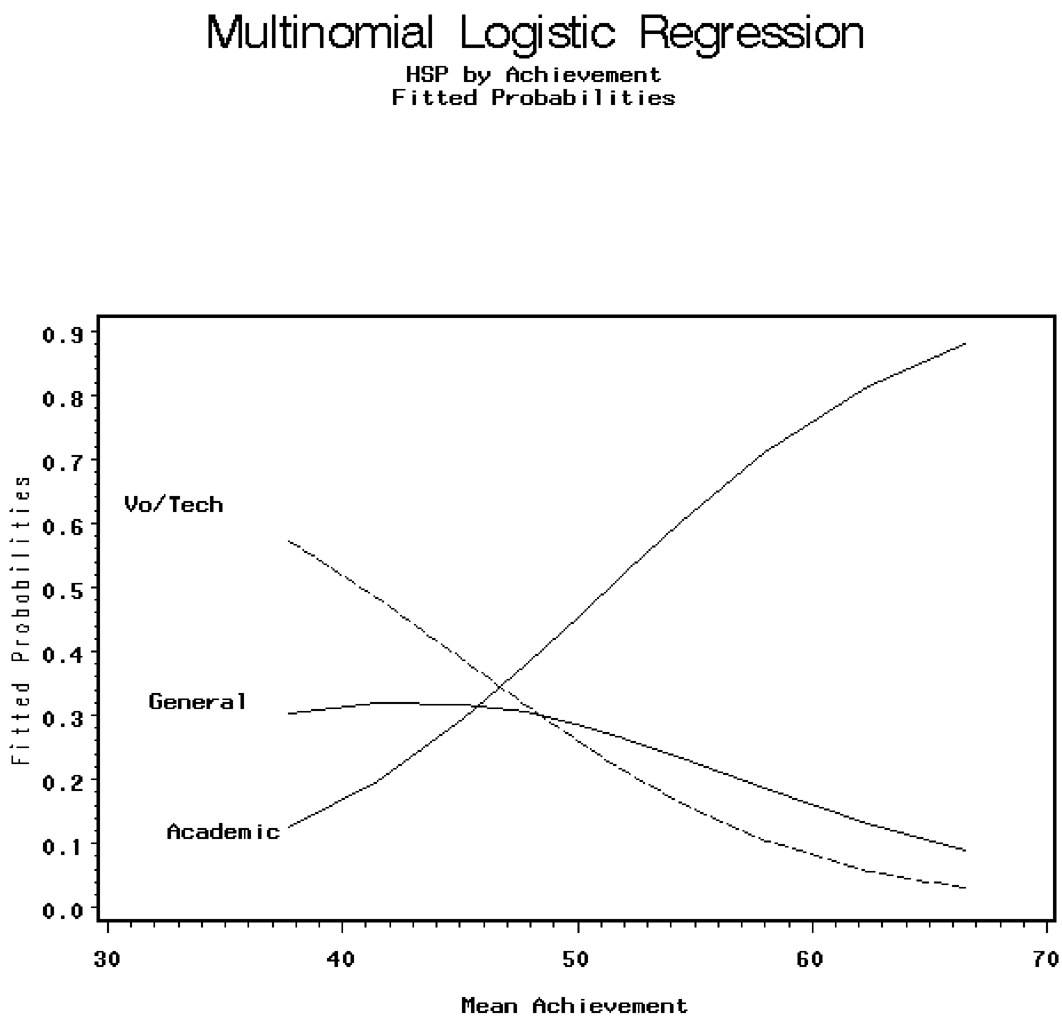
Note:

- The denominator $\Sigma_{h=1}^{J} \exp(\alpha_h + \beta_h x)$ ensures that $\Sigma_{j=1}^{J} \pi_j = 1$.
- $\alpha_J = 0$ and $\beta_J = 0$ (baseline), which is an identification constraint.

Example: High school and beyond

$$\begin{aligned}
\hat{\pi}_{\text{votech}} &= \frac{1}{1 + \exp(-2.90 + .06x) + \exp(-7.94 + .17x)} \\
\hat{\pi}_{\text{general}} &= \frac{\exp(-2.90 + .06x)}{1 + \exp(-2.90 + .06x) + \exp(-7.94 + .17x)} \\
\hat{\pi}_{\text{academic}} &= \frac{\exp(-7.94 + .17x)}{1 + \exp(-2.90 + .06x) + \exp(-7.94 + .17x)}
\end{aligned}$$

which when plotted versus mean achievement scores...


Multinomial Logistic Regression
HSP by Achievement
Fitted Probabilities
Vo/Tech
General
Academic
Fitted Probabilities
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
30
40
50
60
70
Mean Achievement

Statistical Inference

There are 2 kinds of tests we'll talk about here:

1. Test whether an explanatory variable is related to the response variable.
2. Test whether the parameters for two (or more) categories of the response variable are the same.

Both of these tests can be done using either Wald or likelihood ratio (LR) tests. We'll talk about LR tests here; see Long (1997) for the Wald tests.

Test whether an explanatory/predictor variable is not related to the response; that is,

$$H_o : \beta_{k1} = \ldots = \beta_{kJ} = 0$$

for the kth explanatory variable.

Example of LR test: Consider HSB example but now include SES as a nominal variable and then as an ordinal variable.

From the CATMOD output,

Model	-2Log(like)	Δdf	ΔG^2	p-value
achieve, nominal SES	1064.6659	—	—	—
achieve, ordinal SES	1068.2397	2	3.57	.16
achieve	1083.7834	2	15.54	$< .001$

Test whether 2 response categories have the same parameter estimates (i.e., can they be combined?).

If two response categories, j and j', are indistingishable with respect to the variables in the model, then

$$H_o : (\beta_{1j} - \beta_{1j'}) = \ldots = (\beta_{Kj} - \beta_{Kj'}) = 0$$

for the K explanatory variables.

Why don't we have to consider the α's?

There are two LR tests that can be used:

I Fit the model with no restrictions on the parameters, and then fit the model restricting the parameters to be equal.

II Fit a binary logistic regression model to the two response categories in question.

Example: Consider the model with just mean achievement as the explanatory variable.

Method I:

Multinomial/ baseline model	-2L(like)	Δdf	ΔG^2	p-value
No restrictions	1083.7834	—	—	—
$\hat{\beta}_1 = \hat{\beta}_3$	1097.0522	1	13.27	$< .001$

Notes regarding Method I:

- This can be done easily using GENMOD.
- The trick is to create a new variable that is used to impose the equality restriction.

```
data hsblong;
 input student hsp count achieve;
* Create a new dummy variable for equating
  parameters for votech (hsp=3) and general (hsp=1);
 xhsp=0;
 if hsp=2 then xhsp=1;
 datalines;
   1    1    1     41.32000
   1    2    0     41.32000
   1    3    0     41.32000
   2    1    1     45.02000
   2    2    0     45.02000
   2    3    0     45.02000
   3    1    1     34.98000
...
 600    1    0     43.44000
 600    2    0     43.44000
 600    3    1     43.44000
```

```
proc genmod;
 class student hsp;
 model count = student hsp hsp*achieve
  / link=log dist=poi;
 title 'Full Model (no restrictions)';

proc genmod;
 class student hsp;
 model count = student hsp xhsp*achieve
   / link=log dist=poi;
 title 'Equate the slope parameters for
        votech and general';
```

- You can use this method to check whether a sub-set of or specific parameters are equal.
- You can use this trick to see if the parameters for more than two response categories are the same.

Method II: Using the binary logistic regression model to test

$$H_o : (\beta_{1j} - \beta_{2j'}) = \ldots = (\beta_{Kj} - \beta_{Kj'}) = 0$$

for the K explanatory variables.

1. Create a new dataset that only contains the observations from response categories j and j'.
2. Fit the binary logistic regression model to the new dataset.
3. Compute the likelihood ratio statistic that all the slope coefficients (β_k's) are simultaneously equal to 0 — not the intercept term..

Example: We have

LR=13.49 with $df = 1$, $p < .001$.

Notes regarding Methods I and II:

- In this case, both methods give similar results (Method I: LR= 13.27).
- Method I is more flexible in terms of the range of possible tests that can be performed.
- The Method II is much easier. Just how easy this is,

Input

```
data hsb;
 set sasdata.hsb;
 achieve=(RDG+WRTG+MATH+SCI+CIV)/5;
 if hsp=2 then delete;

proc logistic descending;
 model hsp = achieve;
```

Edited Output:

```
The LOGISTIC Procedure

                          Model Information

Data Set                        WORK.HSB
Response Variable               HSP
Number of Response Levels       2
Number of Observations          292
Model                           binary logit
Optimization Technique          Fisher's scoring
```

Response Profile

Ordered Value	HSP	Total Frequency
1	3	147
2	1	145

Probability modeled is HSP=3.

Model Convergence Status

Convergence criterion (GCONV=1E-8) satisfied.

Model Fit Statistics

Criterion	Intercept Only	Intercept and Covariates
AIC	406.784	395.293
SC	410.461	402.647
-2 Log L	404.784	391.293

Testing Global Null Hypothesis: BETA=0

Test	Chi-Square	DF	Pr > ChiSq
Likelihood Ratio	13.4910	1	0.0002
Score	13.2434	1	0.0003
Wald	12.7559	1	0.0004

Baseline/Multinomial Logit model and Grouped Data (Loglinear Model Connection)

When the explanatory/predictor variables are all categorical, the baseline category logit model has an equivalent loglinear model.

Example: Data from Fienberg (1985). In 1963, 2400 men who were rejected for military service because they failed the Armed Forces Qualitifcation Test were interviewed. The data from 2294 of them were as 4–way cross-classification is given below.

The response variable is **E** (respondenent's education) = grammar school, some HS, HS graduate.

The 3 explanatory variables are

R (Race) = White, Black.

A (age) = under 22, 22 or older.

F (father's education) = 1 (grammar school), 2 (some HS), 3 (HS graduate), 4 (not available).

Race	Age	Father's Education	Respondent's Education: Grammar school	some HS	HS graduate
White	< 22	1	39	29	8
		2	4	8	1
		3	11	9	6
		4	48	17	8
	≥ 22	1	231	115	51
		2	17	21	13
		3	18	28	45
		4	197	111	35
Black	< 22	1	19	40	19
		2	5	17	7
		3	2	14	3
		4	49	79	24
	≥ 22	1	110	133	103
		2	18	38	25
		3	11	25	18
		4	178	206	81

Father's education =

1. Grammer school
2. Some high school
3. High school graduate
4. Not available

First, some loglinear modelling (and then the relationship between baseline category logit model and loglinear models).

All models include λ_{ijk}^{RAF} because Race, Age and Father's education are explanatory variables.

Model	*df*	G^2	*p*-value
(RAF,E)	30	254.8	$< .001$
(RAF,EF)	24	162.6	$< .001$
(RAF,EA)	28	242.7	$< .001$
(RAF,ER)	28	152.8	$< .001$
(RAF,EF,EA)	22	151.5	$< .001$
(RAF,EF,ER)	22	46.7	.002
(RAF,EA,ER)	26	142.5	$< .001$
(RAF,EF,EA,ER)	20	36.9	.01
(RAF,EFA,ER)	14	27.9	.01
(RAF,EFR,EA)*	14	18.1	.20
(RAF,EAR,EA)	18	33.2	$< .01$
(RAF,EFA,EFR)	8	9.7	.29

So a good fitting model is

$$\begin{aligned}\log(\mu_{ijkl}) &= \lambda + \lambda_i^R + \lambda_j^A + \lambda_k^F + \lambda_{ij}^{RA} + \lambda_{jk}^{RF} + \lambda_{jk}^{AF} + \lambda_{ijk}^{RAF} \\ &\quad + \lambda_l^E + \lambda_{il}^{RE} + \lambda_{kl}^{FE} + \lambda_{ikl}^{FRE} + \lambda_{jl}^{AE}\end{aligned}$$

Loglinear Model (RAF,EFR,EA):

$$\begin{aligned}\log(\mu_{ijkl}) &= \lambda + \lambda_i^R + \lambda_j^A + \lambda_k^F + \lambda_{ij}^{RA} + \lambda_{jk}^{RF} + \lambda_{jk}^{AF} + \lambda_{ijk}^{RAF} \\ &\quad + \lambda_l^E + \lambda_{il}^{RE} + \lambda_{kl}^{FE} + \lambda_{ikl}^{FRE} + \lambda_{jl}^{AE}\end{aligned}$$

Based on this model, the log odds of grammer school (i.e., $l = 1$) to high school graduate (i.e., $l = 3$) is

$$\begin{aligned}\log(\pi_1/\pi_3) &= \log(\mu_{ijk1}/\mu_{ijk3}) \\ &= \log(\mu_{ijk1}) - \log(\mu_{ijk3}) \\ &= (\lambda_1^E - \lambda_3^E) + (\lambda_{i1}^{RE} - \lambda_{i3}^{RE}) + (\lambda_{j1}^{AE} - \lambda_{j3}^{AE}) \\ &\quad + (\lambda_{k1}^{FE} - \lambda_{k3}^{FE}) + (\lambda_{ik1}^{FRE} - \lambda_{ik3}^{FRE}) \\ &= \alpha + \beta_i^R + \beta_j^A + \beta_k^F + \beta_{ik}^{FR}\end{aligned}$$

where

$\alpha = (\lambda_1^E - \lambda_3^E)$

$\beta_i^R = (\lambda_{i1}^{RE} - \lambda_{i3}^{RE})$

$\beta_j^A = (\lambda_{j1}^{AE} - \lambda_{j3}^{AE})$

$\beta_k^F = (\lambda_{k1}^{FE} - \lambda_{k3}^{FE})$

$\beta_{ik}^{RF} = (\lambda_{ik1}^{RFE} - \lambda_{ik3}^{RFE})$

The logit model for any other pair of categories of the respondenent's education is found in an analogous way (i.e., simply take the difference between appropriate λ's from the loglinear model).

Estimated parameters of the baseline categories logit model with HS graduate at the baseline:

			Grammar vs HS grad		Some HS vs HS grad	
			Value	$\exp(\hat{\beta})$	Value	$\exp(\hat{\beta})$
$\hat{\alpha}$			.73		.90	
$\hat{\beta}_i^R$	white		.98		.12	
	black		.00		.00	
$\hat{\beta}_j^A$	< 22		.20	1.22	.44	1.55
	≥ 22		.00		.00	
$\hat{\beta}_k^F$	1		-.71		-.63	
	2		-1.10		-.47	
	3		-1.25		-.39	
	4		.00		.00	
$\hat{\beta}_{ik}^{RF}$	white	1	.49	1.63	.43	1.54
		2	-.23	.79	.09	1.09
		3	-1.06	.35	-1.05	.35
		4	.00		.00	
	black	1	.00		.00	
		2	.00		.00	
		3	.00		.00	
		4	.00		.00	

F = Father's education = 1 grammar, 2 some HS, 3 HS graduate, 4 not available.

Some Intpretation:

- Age: The odds that someone younger than 22 has had some high school (versus graduating) are 1.55 times larger than the odds for a person older than 22. An older person has had more time to complete HS.
- $1/.35 = 2.86 \Longrightarrow$ Given that the father graduated from high school, the odds that a black completes some high school (versus graduates) are 2.86 times time larger than the odds that a white completes some high school (versus graduates).

Final few remarks regarding baseline category model

- With the baseline category logit model, there is a single global fit statistic, which is valid if sample size is large enough (eg. for grouped data).
- This model can be used when the categories of the response variable are ordered, but it may not be the best model for the case of ordinal responses.
- The explanatory variable has the same value regardless of which 2 categories/levels of the response variable that are being compared.
- The model and interpretation can be very complex because for each way of forming odds, there are different parameters.
- The multinomial logit model described here can also be derived as a choice model based on random utilities.
- Bock's nominal response (IRT) model for polytomous items

$$P(Y = j|\theta) = \frac{\exp(\alpha_j + \beta_j\theta)}{\Sigma_{h=1}^{J} \exp(\alpha_h + \beta_h\theta)}$$

  where θ is an unobserved explanatory variable.

Conditional Logit Model

In Psychology, this is either Bradley & Terry (1952) or the Luce (1959) choice model. In business/economics, this is McFadden's (1974) conditional logit model.

Situation: Individuals are given a set of possible choices, which differ on certain attributes. We would like to model/predict the probability of choices using the attributes of the choices as explanatory/predictor variables.

Examples:

- Subjects are given 8 chocolate candies and asked which one they like the best (SAS Logistic Regression examples, 1995; Kuhfeld; 2001). The explanatory variables are
  - Type of chocolate: milk or dark
  - Texture: hard or soft
  - Include nuts: nuts or no nuts
- Inidividuals must choose which of 5 brands of a product that they prefer (SAS Logistic Regression examples, 1995; Kuhfeld; 2001). The explanatory variable is the price of the product. The company presents different combinations of prices for the different brands to see how much of an effect this has on choice behavior.
- The classic example: choice of mode of transportation (eg, train, bus, car). Characteristics or attributes of these include time waiting, how long it takes to get to work, and cost.

The conditional logit model:

- The coefficients of the explanatory variables are the same over the categories (choices) of the response variable.
- The values of the explanatory variables differ over the outcomes (and possibly over individuals).

$$\pi_j(x_{ij}) = \frac{\exp[\alpha + \beta x_{ij}]}{\Sigma_{j \epsilon C_i} \exp[\alpha + \beta x_{ij}]}$$

where

x_{ij} is the value of the explanatory variable for individual i and response choice j.

The summation in the denominator is over response options/choices that individual i is given.

Properties of this model:

- The odds that individual i chooses option j versus k is a function of the difference between x_{ij} and x_{ik}:

$$\log\left(\frac{\pi_j(x_{ij})}{\pi_k(x_{ik})}\right) = \beta(x_{ij} - x_{ik})$$

- The odds of choosing j versus k does not depend on any of the other options in the choice set or the other options' values on the attribute variables.

  Property of "Independence from Irrelevant Alternatives".

- The multinomial/baseline model can be written in the same form as the conditional logit model model (see Agresti (90), p 316-317).
- This model can incorporate attributes or characteristics of the decision maker/individual.
- It can be written as a proportional hazard model.

Examples:

1. Three examples that only include attributes of the response alternatives.
2. An example that includes both attributes of the response alternatives and characteristics of the individual ("mixed model").

Example 1: chocolates

The model that was fit is

$$\pi_j(c_j, t_j, n_j) = \frac{\exp[\alpha + \beta_1 c_j + \beta_2 t_j + \beta_3 n_j]}{\Sigma_{h=1}^{8}(\exp[\alpha + \beta_1 c_h + \beta_2 t_h + \beta_3 n_h])}$$

where

- Type of chocolate is dummy coded:

$$c_j = \begin{cases} 1 & \text{if milk} \\ 0 & \text{if dark} \end{cases}$$

- Texture is dummy coded:

$$t_j = \begin{cases} 1 & \text{if hard} \\ 0 & \text{if soft} \end{cases}$$

- Nuts is dummy coded:

$$n_j = \begin{cases} 1 & \text{if no nuts} \\ 0 & \text{if nuts} \end{cases}$$

Or in terms of Odds:

$$\frac{\pi_j(c_j, t_j, n_j)}{\pi_k(c_k, t_k, n_k)} = \exp[\beta_1(c_j - c_k)]\exp[\beta_2(t_j - t_k)]\exp[\beta_3(n_j - n_k)]$$

parameter	*df*	value	ASE	Wald	*p*	$\exp\beta$	
α	1	-2.88	1.03	7.78	.01	—	
Type of chocolate							
milk	1	-1.38	.79	3.07	.08	.25	or $(1/.25) = 4.00$
dark	0	0.00					
Texture							
hard	1	2.20	1.05	4.35	.04	9.00	
soft	0	0.00					
Nuts							
no nuts	1	-.85	.69	1.51	.22	.43	or $(1/.43) = 2.33$
nuts	0	0.00					

Use $\exp\beta$ for interpretation.

The predicted probabilities.

```
Obs    drk     sft     nts      phat
 1     dark    hard    nuts    0.50400
 2     dark    hard    no n    0.21600
 3     milk    hard    nuts    0.12600
 4     dark    soft    nuts    0.05600
 5     milk    hard    no n    0.05400
 6     dark    soft    no n    0.02400
 7     milk    soft    nuts    0.01400
 8     milk    soft    no n    0.00600
```

Estimation of the model:

1. SAS Logistic Regression Examples (1995) and Kuhfeld (2001; http://www.sas.com/service/techsup/tnote/tnote_stat.html) describes how this can be done using proc PHREG (proportional hazard regression), which is related to Poisson regression. The "trick" here is to dummy code for "time" so that the non-selected category is 1 and the choosen is 0.

2. The model can be fit as a Poisson regression model using GENMOD by appropriately arranging the data.

The Data file:

```
data chocs;
   title 'Chocolate Candy Data';
   input subj choose dark soft nuts @@;
   t=2-choose;                                       * Needed when using PROC PHREG;
   if dark=1 then drk='dark'; else drk='milk';       * Needed for printing out table;
   if soft=1 then sft='soft'; else sft='hard';       *  of predicted probabilities;
   if nuts=1 then nts='nuts'; else nts='no nuts';
   datalines;
 1 0 0 0 0    1 0 0 0 1   1 0 0 1 0    1 0 0 1 1
 1 1 1 0 0    1 0 1 0 1   1 0 1 1 0    1 0 1 1 1
 2 0 0 0 0    2 0 0 0 1   2 0 0 1 0    2 0 0 1 1
 2 0 1 0 0    2 1 1 0 1   2 0 1 1 0    2 0 1 1 1
 3 0 0 0 0    3 0 0 0 1   3 0 0 1 0    3 0 0 1 1
 3 0 1 0 0    3 0 1 0 1   3 1 1 1 0    3 0 1 1 1
 4 0 0 0 0    4 0 0 0 1   4 0 0 1 0    4 0 0 1 1
 4 1 1 0 0    4 0 1 0 1   4 0 1 1 0    4 0 1 1 1
 5 0 0 0 0    5 1 0 0 1   5 0 0 1 0    5 0 0 1 1
 5 0 1 0 0    5 0 1 0 1   5 0 1 1 0    5 0 1 1 1
 6 0 0 0 0    6 0 0 0 1   6 0 0 1 0    6 0 0 1 1
 6 0 1 0 0    6 1 1 0 1   6 0 1 1 0    6 0 1 1 1
 7 0 0 0 0    7 1 0 0 1   7 0 0 1 0    7 0 0 1 1
 7 0 1 0 0    7 0 1 0 1   7 0 1 1 0    7 0 1 1 1
 8 0 0 0 0    8 0 0 0 1   8 0 0 1 0    8 0 0 1 1
 8 0 1 0 0    8 1 1 0 1   8 0 1 1 0    8 0 1 1 1
```

```
 9 0 0 0 0    9 0 0 0 1   9 0 0 1 0    9 0 0 1 1
 9 0 1 0 0    9 1 1 0 1   9 0 1 1 0    9 0 1 1 1
10 0 0 0 0   10 0 0 0 1  10 0 0 1 0   10 0 0 1 1
10 0 1 0 0   10 1 1 0 1  10 0 1 1 0   10 0 1 1 1
```

Using SAS/GENMOD:

```
proc genmod data=chocs;
 class subj dark soft nuts;
 model choose = dark soft nuts /link=log dist=poi obstats;
 output out=fitted pred=phat;
 title 'Conditional logit model using GENMOD';

data subset;
 merge chocs fitted;
 if subj>1 then delete;

proc sort;
 by descending phat;

proc print;
 var drk sft nts phat;
 title 'Predicted probabilities for different chocolates';
```

Output from GENMOD

```
       Model Information

Data Set              WORK.CHOCS
Distribution             Poisson
Link Function                Log
Dependent Variable        choose
Observations Used             80


         Class Level Information

Class      Levels    Values

subj           10    1 2 3 4 5 6 7 8 9 10
dark            2    0 1
soft            2    0 1
nuts            2    0 1


           Criteria For Assessing Goodness Of Fit

Criterion                 DF           Value        Value/DF

Deviance                  76         28.7270          0.3780
Scaled Deviance           76         28.7270          0.3780
Pearson Chi-Square        76         66.7195          0.8779
Scaled Pearson X2         76         66.7195          0.8779
Log Likelihood                      -24.3635


Algorithm converged.
```

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr > ChiSq
Intercept		1	-2.8824	1.0334	-4.9078	-0.8570	7.78	0.0053
dark	0	1	-1.3863	0.7906	-2.9358	0.1632	3.07	0.0795
dark	1	0	0.0000	0.0000	0.0000	0.0000	.	.
soft	0	1	2.1972	1.0541	0.1312	4.2632	4.35	0.0371
soft	1	0	0.0000	0.0000	0.0000	0.0000	.	.
nuts	0	1	-0.8473	0.6901	-2.1998	0.5052	1.51	0.2195
nuts	1	0	0.0000	0.0000	0.0000	0.0000	.	.
Scale		0	1.0000	0.0000	1.0000	1.0000		

Using PROC PHREG.

But first, what's a proportional hazard regression?

- It's typically used for modeling surival data; that is, modeling the time until death (or other event of interest).
- It's equivalent to a Poisson regression for the number of deaths and to a negative expontential for survival times.
- For more details see Agresti (1990).

Using SAS PROC PHREG: input

```
proc phreg data=chocs outest=betas;
 strata subj;
  model t*choose(0)=dark soft nuts;
  title 'Conditional Logit model fit using PROC PHREG';
run;
```

Output:

```
The PHREG Procedure

         Model Information

Data Set                 WORK.CHOCS
Dependent Variable       t
Censoring Variable       choose
Censoring Value(s)       0
Ties Handling            BRESLOW
```

Summary of the Number of Event and Censored Values

Stratum	subj	Total	Event	Censored	Percent Censored
1	1	8	1	7	87.50
2	2	8	1	7	87.50
3	3	8	1	7	87.50
4	4	8	1	7	87.50
5	5	8	1	7	87.50
6	6	8	1	7	87.50
7	7	8	1	7	87.50
8	8	8	1	7	87.50
9	9	8	1	7	87.50
10	10	8	1	7	87.50
Total		80	10	70	87.50

Convergence Status

Convergence criterion (GCONV=1E-8) satisfied.

Model Fit Statistics

Criterion	Without Covariates	With Covariates
-2 LOG L	41.589	28.727
AIC	41.589	34.727
SBC	41.589	35.635

Testing Global Null Hypothesis: BETA=0

Test	Chi-Square	DF	Pr > ChiSq
Likelihood Ratio	12.8618	3	0.0049
Score	11.6000	3	0.0089
Wald	8.9275	3	0.0303

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Chi-Square	Pr > ChiSq	Hazard Ratio
dark	1	1.38629	0.79057	3.0749	0.0795	4.000
soft	1	-2.19722	1.05409	4.3450	0.0371	0.111
nuts	1	0.84730	0.69007	1.5076	0.2195	2.333

Example 2: Five brands that differ in terms of price where price is manipulated. For each of the 8 combinations of brand and price included in the study, 100 individuals made their choice.

In all models that we fit, we assume (i.e., fit a parameter) for brand preference.

The two models that are fit:

1. The effect of price does not depend on brand.
2. The effect of price depends on the brand (i.e. the strength of brand loyalty depends on price).

Complex model: $G^2 = 2782.0879$

Simpler model: $G^2 = 2782.4901$

LR statistic for testing whether effect of price depends on brand:

$$G^2 = 2782.4901 - 2782.0879 = .4022, \qquad df = 3, \qquad p = .94$$

So let's look at simpler model…

$$\pi_j(b_{1j}, b_{2j}, b_{3j}, b_{4j}, p_j) = \frac{\exp[\alpha + \beta_1 b_{1j} + \beta_2 b_{2j} + \beta_3 b_{3j} + \beta_4 b_{4j} + \beta_5 p_j]}{\Sigma_{h=1}^{5} \exp[\alpha + \beta_1 b_{1h} + \beta_2 b_{2h} + \beta_3 b_{3h} + \beta_4 b_{4h} + \beta_5 p_h]}$$

where

- Brands are dummy coded. Eg,

$$b_{1j} = \begin{cases} 1 & \text{if brand is 1} \\ 0 & \text{otherwise} \end{cases}$$

  Note: for the 5$^{\text{th}}$ brand, $b_{1j} = b_{2j} = b_{3j} = b_{4j} = 0$.

- Price is a numerical variable, p_j.

Or in terms of odds:

$$\begin{aligned} \frac{\pi_j(b_{1j}, b_{2j}, b_{3j}, b_{4j}, p_j)}{\pi_k(b_{1k}, b_{2k}, b_{3k}, b_{4k}, p_k)} &= \exp[\beta_1(b_{1j} - b_{1k})] \exp[\beta_2(b_{2j} - b_{2k})] \\ &\quad \exp[\beta_3(b_{3j} - b_{3k})] \exp[\beta_4(b_{4j} - b_{4k})] \\ &\quad \exp[\beta_5(p_j - p_k)] \end{aligned}$$

Estimated parameters for the common price model:

Variable		DF	Parameter Estimate	Standard Error	Chi-Square	p	$\exp \hat{\beta}$
brand1	β_1	1	0.66727	0.12305	29.4065	$< .0001$	1.95
brand2	β_2	1	0.38503	0.12962	8.8235	0.0030	1.47
brand3	β_3	1	−0.15955	0.14725	1.1740	0.2786	.85
brand4	β_4	1	0.98964	0.11720	71.2993	$< .0001$	2.69
brand5	—	0	0	.	.	.	1.00
price	β_5	1	0.14966	0.04406	11.5379	0.0007	1.16

- Which brand is the most preferred?
- Which brand is least preferred?
- What is the effect of price?

  How would you interpret $\exp[.1497] = 1.16$?

Estimating the common price effect and the price $\times$ brand interaction model using SAS:

GENMOD: as a Poisson regression model.

PHREG: as a proportional harzard model.

First the "raw" data:

```
data brands;
 title 'Brand Choice Data';
 input p1-p5 f1-f5;
 datalines;
5.99 5.99 5.99 5.99 4.99  12 19 22 33 14
5.99 5.99 3.99 3.99 4.99  34 26  8 27  5
5.99 3.99 5.99 3.99 4.99  13 37 15 27  8
5.99 3.99 3.99 5.99 4.99  49  1  9 37  4
3.99 5.99 5.99 3.99 4.99  31 12  6 18 33
3.99 5.99 3.99 5.99 4.99   4 29 16 42  9
3.99 3.99 5.99 5.99 4.99  37 10  5 35 13
3.99 3.99 3.99 3.99 4.99  16 14  5 51 14
```

Format of data needed for input to GENMOD:

```
data brands2;
 input combo brand price choice @@;
 datalines;
1 1 5.99 12   1 2 5.99  0   1 3 5.99  0   1 4 5.99  0   1 5 4.99  0
1 1 5.99  0   1 2 5.99 19   1 3 5.99  0   1 4 5.99  0   1 5 4.99  0
1 1 5.99  0   1 2 5.99  0   1 3 5.99 22   1 4 5.99  0   1 5 4.99  0
1 1 5.99  0   1 2 5.99  0   1 3 5.99  0   1 4 5.99 33   1 5 4.99  0
1 1 5.99  0   1 2 5.99  0   1 3 5.99  0   1 4 5.99  0   1 5 4.99 14
2 1 5.99 34   2 2 5.99  0   2 3 3.99  0   2 4 3.99  0   2 5 4.99  0
2 1 5.99  0   2 2 5.99 26   2 3 3.99  0   2 4 3.99  0   2 5 4.99  0
2 1 5.99  0   2 2 5.99  0   2 3 3.99  8   2 4 3.99  0   2 5 4.99  0
2 1 5.99  0   2 2 5.99  0   2 3 3.99  0   2 4 3.99 27   2 5 4.99  0
2 1 5.99  0   2 2 5.99  0   2 3 3.99  0   2 4 3.99  0   2 5 4.99  5
etc.
```

PROC GENMOD commands:

```
proc genmod;
 class combo brand ;
 model choice = combo brand price /link=log dist=poi;
title 'Brands Model 1 ';

proc genmod;
 class combo brand ;
 model choice = combo brand brand*price /link=log dist=poi;
title 'Brands Model 2 ';
run;
```

Some Edited output from Brands Model 1 (common price effect):

```
The GENMOD Procedure

        Model Information

Data Set                    WORK.BRANDS2
Distribution                     Poisson
Link Function                        Log
Dependent Variable                choice
Observations Used                    200

       Class Level Information

Class      Levels    Values

combo           8    1 2 3 4 5 6 7 8
brand           5    1 2 3 4 5


           Criteria For Assessing Goodness Of Fit
Criterion                DF           Value        Value/DF

Deviance                187       2782.4901         14.8796
Scaled Deviance         187       2782.4901         14.8796
Pearson Chi-Square      187       4235.1363         22.6478
Scaled Pearson X2       187       4235.1363         22.6478
Log Likelihood                     383.9789
```

```
Algorithm converged.
```

Analysis Of Parameter Estimates

Parameter		DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square
Intercept		1	0.3039	0.2249	-0.1368	0.7446	1.83
combo	1	1	-0.2616	0.1610	-0.5772	0.0539	2.64
combo	2	1	-0.1370	0.1478	-0.4268	0.1527	0.86
combo	3	1	-0.1136	0.1460	-0.3997	0.1725	0.61
combo	4	1	-0.1817	0.1520	-0.4797	0.1163	1.43
combo	5	1	-0.0951	0.1447	-0.3787	0.1885	0.43
combo	6	1	-0.1644	0.1503	-0.4590	0.1302	1.20
combo	7	1	-0.1417	0.1483	-0.4322	0.1489	0.91
combo	8	0	0.0000	0.0000	0.0000	0.0000	.
brand	1	1	0.6673	0.1230	0.4261	0.9084	29.41
brand	2	1	0.3850	0.1296	0.1310	0.6391	8.82
brand	3	1	-0.1595	0.1472	-0.4481	0.1291	1.17
brand	4	1	0.9896	0.1172	0.7599	1.2194	71.30
brand	5	0	0.0000	0.0000	0.0000	0.0000	.
price		1	0.1497	0.0441	0.0633	0.2360	11.54
Scale		0	1.0000	0.0000	1.0000	1.0000	

- Intercept and combo are nusiance parameters.
- Brand 1 through Brand 5 are $\beta_1 - \beta_4$.
- Price is β_5.

Now using PHREG.

The following program is basically from the SAS Logistic Regression (1995) book, which is also pretty much the same as the program in Kuhfeld (2001).

This section puts the data in the format needed for PROC PHREG:

```
data brands3;
 set brands;
 drop p1-p5 f1-f5;

* Define arrays for variables of the orginal data set;
 array p[5] p1-p5;                /* Array for prices*/
 array f[5] f1-f5;                /* Array for frequencies*/

* Define arrays for design matrices in new data set;
 array pb[5] price1-price5;       /* Array for prices*/
 array brand[5] brand1-brand5;  /* Aarray for brands*/;

* Initialize brand and brand by price design matrices;
 do j=1 to 5;               /* 5=number of choice options*/
  brand[j]=0;
  pb[j]=0;
 end;

* Count the total number of choices;
 nobs = sum(of f1-f5);
```

```
* Store choice set number to stratify;

 ch_set=_n_;

* Create design matrix;
 do j=1 to 5;
  price = p[j];
  brand[j]=1;
  pb[j] = price;

  * Output number of times each brand choosen;
  freq = f[j];
  choose=1;
  t = 1;    /* choice occurs at time 1 */
  output;

  * Output number of times each brand was not choosen;
  freq = nobs-f[j];
  choose =0;
  t = 2;    /* NON choice occurs at time 2 */
  output;

  * Set up for next alternative;
  brand[j] = 0;
  pb[j] = 0;
 end;
run;
```

The data looks like:

Obs	price1	price2	price3	price4	price5	brand1	brand2	brand3	brand4	brand5	j	nobs	ch_set	price	freq	choose	t
1	5.99	0.00	0.00	0.00	0.00	1	0	0	0	0	1	100	1	5.99	12	1	1
2	5.99	0.00	0.00	0.00	0.00	1	0	0	0	0	1	100	1	5.99	88	0	2
3	0.00	5.99	0.00	0.00	0.00	0	1	0	0	0	2	100	1	5.99	19	1	1
4	0.00	5.99	0.00	0.00	0.00	0	1	0	0	0	2	100	1	5.99	81	0	2
5	0.00	0.00	5.99	0.00	0.00	0	0	1	0	0	3	100	1	5.99	22	1	1
6	0.00	0.00	5.99	0.00	0.00	0	0	1	0	0	3	100	1	5.99	78	0	2
7	0.00	0.00	0.00	5.99	0.00	0	0	0	1	0	4	100	1	5.99	33	1	1
8	0.00	0.00	0.00	5.99	0.00	0	0	0	1	0	4	100	1	5.99	67	0	2
9	0.00	0.00	0.00	0.00	4.99	0	0	0	0	1	5	100	1	4.99	14	1	1
10	0.00	0.00	0.00	0.00	4.99	0	0	0	0	1	5	100	1	4.99	86	0	2
11	5.99	0.00	0.00	0.00	0.00	1	0	0	0	0	1	100	2	5.99	34	1	1
12	5.99	0.00	0.00	0.00	0.00	1	0	0	0	0	1	100	2	5.99	66	0	2
13	0.00	5.99	0.00	0.00	0.00	0	1	0	0	0	2	100	2	5.99	26	1	1
14	0.00	5.99	0.00	0.00	0.00	0	1	0	0	0	2	100	2	5.99	74	0	2
15	0.00	0.00	3.99	0.00	0.00	0	0	1	0	0	3	100	2	3.99	8	1	1

The PHREG commands to fit the two models:

```
proc phreg data=brands3;
 strata ch_set;
 model t*choose(0)=brand1 brand2 brand3 brand4 brand5 price;
 freq freq;
 title 'PHREG:  Discrete choice with common price effect';
```

```
proc phreg data=brands3;
 strata ch_set;
 model t*choose(0)=brand1-brand5 price1-price5;
 freq freq;
 title 'PHREG:  Discrete choice with brand by price effect';
run;
```

A little edited output:

```
The PHREG Procedure

          Model Information

Data Set                 WORK.BRANDS3
Dependent Variable       t
Censoring Variable       choose
Censoring Value(s)       0
Frequency Variable       freq
Ties Handling             BRESLOW
```

Summary of the Number of Event and Censored Values

Stratum	ch_set	Total	Event	Censored	Percent Censored
1	1	500	100	400	80.00
2	2	500	100	400	80.00
3	3	500	100	400	80.00
4	4	500	100	400	80.00
5	5	500	100	400	80.00
6	6	500	100	400	80.00
7	7	500	100	400	80.00
8	8	500	100	400	80.00
Total		4000	800	3200	80.00

Convergence Status

Convergence criterion (GCONV=1E-8) satisfied.

Model Fit Statistics

Criterion	Without Covariates	With Covariates
-2 LOG L	9943.373	9793.486
AIC	9943.373	9803.486
SBC	9943.373	9826.909

The PHREG Procedure

Analysis of Maximum Likelihood Estimates

Variable	DF	Parameter Estimate	Standard Error	Chi-Square	Pr > ChiSq	Hazard Ratio
brand1	1	0.66727	0.12305	29.4065	<.0001	1.949
brand2	1	0.38503	0.12962	8.8235	0.0030	1.470
brand3	1	-0.15955	0.14725	1.1740	0.2786	0.853
brand4	1	0.98964	0.11720	71.2993	<.0001	2.690
brand5	0	0	.	.	.	.
price	1	0.14966	0.04406	11.5379	0.0007	1.161

Example 3: From Powers & Xie (2000). $n = 152$ respondents.

The Response variable is mode of transportation:
$j = 1$ for train, 2 for bus, and 3 for car.
Explanatory Variables are:

t_{ij} = time waiting in Terminal.

v_{ij} = time spent in the Vehicle.

c_{ij} = Cost of time spent in vehicle.

g_{ij} = Generalized cost measure = $c_{ij} + v_{ij}(\text{value}_{ij})$ where value equals subjective value of respondent's time for each mode of transportation.

The multinomial logit model that appears to fit the data is

$$\pi_{ij} = \frac{\exp[\beta_1 t_{ij} + \beta_2 v_{ij} + \beta_3 c_{ij} + \beta_4 g_{ij}]}{\Sigma_{h=1}^{3} \exp[\beta_1 t_{ih} + \beta_2 v_{ih} + \beta_3 c_{ih} + \beta_4 g_{ih}]}$$

The odds of choosing mode j versus mode k for individual i,

$$\frac{\pi_{ij}}{\pi_{ik}} = \exp[\beta_1(t_{ij} - t_{ik})]\exp[\beta_2(v_{ij} - v_{ik})]\exp[\beta_3(c_{ij} - c_{ik})]\exp[\beta_4(g_{ij} - g_{ik})]$$

The odds of choosing mode j versus mode k for individual i,

$$\frac{\pi_{ij}}{\pi_{ik}} = \exp[\beta_1(t_{ij} - t_{ik})] \exp[\beta_2(v_{ij} - v_{ik})] \exp[\beta_3(c_{ij} - c_{ik})] \exp[\beta_4(g_{ij} - g_{ik})]$$

Variable	Parameter	Value	ASE	Wald	p-value	e^β	$1/e^\beta$
terminal, t_{ij}	β_1	$-.002$	.007	.098	.75	.99	1.002
vehicle, v_{ij}	β_2	$-.435$	.133	10.75	.001	.65	1.55
cost, c_{ij}	β_3	$-.077$	.019	15.93	$< .001$	.03	1.08
generalized cost, g_{ij}	β_4	.431	.133	10.48	.001	1.54	.65

Odds of choosing a particular mode of transportation decreases as

- Time waiting in terminal increases.
- Time spent in vehicle increases.
- Cost increases.

Odds of choosing a particular model of transportation increases as

- Generalized cost (value of individual's time) increases

The Mixed Model

The conditional multinomial model that incorporates attributes of the categories (choices) and of the decision maker.

This model is a combination of the multinomial and conditional multinomial modela.

Suppose

- Response variable Y has J categories/levels.
- Explanatory variables

  x_i that is a measure of an attribute of individual i

  w_j that is a measure of an attribute of alternative j.

  z_{ij} that is a measure of an attribute of alternative j for individual i.

The "Mixed" Model:

$$\pi_j(x_i, w_j, z_{ij}) = \frac{\exp[\alpha_j + \beta_{1j}x_i + \beta_2 w_j + \beta_3 z_{ij}]}{\Sigma_{h=1}^{J} \exp[\alpha_h + \beta_{1h}x_i + \beta_2 w_h + \beta_3 z_{ih}]}$$

The odds of individual i choosing category j versus category k,

$$\begin{aligned}\frac{\pi_j(x_i, w_j, z_{ij})}{\pi_k(x_i, w_k, z_{ik})} &= \exp[\alpha_j - \alpha_k] \exp[(\beta_{1j} - \beta_{1k})x_i] \\ &\quad \exp[\beta_2(w_j - w_k)] \exp[\beta_3(z_{ij} - z_{ik})]\end{aligned}$$

Transportation Example (continued)...

From Powers & Xie (2000). $n = 152$ respondents.

The Response variable is mode of transportation:
$j = 1$ for train, 2 for bus, and 3 for car.

Explanatory Variables are:

t_{ij} = time waiting in Terminal.

v_{ij} = time spent in the Vehicle.

c_{ij} = Cost of time spent in vehicle.

g_{ij} = Generalized cost measure = $c_{ij} + v_{ij}(\text{value}_{ij})$ where value equals subjective value of respondent's time for each mode of transportation.

h_i = Household income.

The mixed model that appears to fit the data is

$$\pi_{ij} = \frac{\exp[\beta_1 t_{ij} + \beta_2 v_{ij} + \beta_3 c_{ij} + \beta_4 g_{ij} + \alpha_j + \beta_{5j} h_i]}{\Sigma_{h=1}^{3} \exp[\beta_1 t_{ih} + \beta_2 v_{ih} + \beta_3 c_{ih} + \beta_4 g_{ih} + \alpha_h + \beta_{5h} h_i]}$$

The odds of choosing mode j versus mode k for individual i,

$$\begin{aligned}\frac{\pi_{ij}}{\pi_{ik}} &= \exp[\beta_1(t_{ij} - t_{ik})] \exp[\beta_2(v_{ij} - v_{ik})] \exp[\beta_3(c_{ij} - c_{ik})] \exp[\beta_4(g_{ij} - g_{ik})] \\ &\quad \exp[(\alpha_j - \alpha_k)] \exp[(\beta_{5j} - \beta_{5k})h_i]\end{aligned}$$

The odds of choosing mode j versus mode k for individual i,

$$\frac{\pi_{ij}}{\pi_{ik}} = \exp[\beta_1(t_{ij} - t_{ik})] \exp[\beta_2(v_{ij} - v_{ik})] \exp[\beta_3(c_{ij} - c_{ik})] \exp[\beta_4(g_{ij} - g_{ik})] \exp[(\alpha_j - \alpha_k)] \exp[(\beta_{5j} - \beta_{5k})h_i]$$

Parameter Estimates:

Variable	Parameter	Value	ASE	Wald	p-value	e^β	$1/e^\beta$
Terminal, t_{ij}	β_1	$-.074$	.017	19.01	$< .001$	.93	1.08
Vehicle, v_{ij}	β_2	$-.619$	.152	16.54	$< .001$	.54	1.86
Cost, c_{ij}	β_3	$-.096$	.022	19.02	$< .001$	.91	1.10
Generalized cost, g_{ij}	β_4	.581	.150	15.08	$< .001$	1.79	.56
Bus							
Intercept,	α_1	-2.108	.730	6.64	.01		
Income, h_i	β_{51}	.031	.021	1.97	.16	1.03	.97
Car							
Intercept	α_2	-6.147	1.029	35.70	$< .001$		
Income, h_i	β_{52}	.048	.023	7.19	.01	1.05	.95

Effect of household income:

- The odds of choosing a bus versus a train given household income increases from h_i to $h_i + 100$ unit is $\exp(100(.031)) = 22.2$ times larger.
- The odds of choosing a car versus a train given household income increases from h_i to $h_i + 100$ unit is $\exp(100(.048)) = 121.5$ times larger.

- The odds of choosing a car versus a bus given household income increases from h_i to $h_i + 100$ unit is $\exp(100(.048 - .031)) = \exp(1.7) = 5.5$ times larger.

Logit Models for ordinal responses

Situation: Polytomous response and categories are ordered.

The logit model for this situation

- Use the ordering of the categories in forming logits.
- Yield simpler models with simpler interpretations than nominal model.
- Is more powerful than nominal models.

Outline:

1. Cumulative logit model, or the "proportional odds" model.
2. Adjacent categories logit model.
3. Continuation ratio logits.

Cumulative Logit Model

Forming logits or how to dichotomize categories of Y such that we incorporate the ordinal information.

Use Cumulative Probabilities:

$Y = 1, 2, \ldots, J$ and order is relavant.

$\{\pi_1, \pi_2, \ldots, \pi_J\}$.

$P(Y \leq j) = \pi_1 + \ldots + \pi_j = \Sigma_{h=1}^{j} \pi_h$ for $j = 1, \ldots, J-1$.

"Cumulative logits"

$$\log\left(\frac{P(Y \leq j)}{P(Y > j)}\right) = \log\left(\frac{P(Y \leq j)}{1 - P(Y \leq j)}\right)$$
$$= \log\left(\frac{\pi_1 + \ldots + \pi_j}{\pi_{j+1} + \ldots + \pi_J}\right) \qquad \text{for} \quad j = 1, \ldots, J-1$$

The "**Proportional Odds Model**"

$$\text{logit}[P(Y \leq j)] = \log\left(\frac{P(Y \leq j)}{P(Y > j)}\right) = \alpha_j + \beta x \qquad \text{for} \quad j = 1, \ldots, J-1$$

- α_j (intercepts) can differ.
- β (slope) is constant.
  - The effect of x is the same for all $J-1$ ways to collapse Y into dichotomous outcomes.
  - A single parameter describes the effect of x on Y (versus $J-1$ in the baseline model).

- Interpretation in terms of odds ratios.

For a given level of Y (say $Y = j$)

$$\frac{P(Y \leq j|X = x_2)/P(Y > j|X = x_2)}{P(Y \leq j|X = x_1)/P(Y > J|X = x_1)} = \frac{P(Y \leq j|x_2)P(Y > j|x_1)}{P(Y \leq j|x_1)P(Y > j|x_2)}$$

$$= \exp(\alpha_j + \beta x_2)/\exp(\alpha_j + \beta x_1)$$

$$= \exp[\beta(x_2 - x_1)]$$

or log odds ratio $= \beta(x_2 - x_1)$.

The odds ratio is proportional to the difference (distance) between x_1 and x_2 (this is sometimes referred to as "difference model").

Since the proportionality $= \beta$ is constant, this model is called the "Proportional Odds Model".

- Note that the cumulative probabilities are given by

  $$P(Y \leq j) = \frac{\exp(\alpha_j + \beta x)}{1 + \exp(\alpha_j + \beta x)}$$

  Since β is constant, curves of cumulative probabilities plotted against x are parallel.

- We can compute the probability of being in category j by taking differences between the cumulative probabilities.

  $$P(Y = j) = P(Y \leq j) - P(Y \leq j - 1) \qquad \text{for} \quad j = 2, \ldots, J$$

  and

  $$P(Y = 1) = P(Y \leq 1)$$

Since β is constant, these probabilities are guaranteed to be non-negative.

- In fitting this model to data, it must be simultaneous.

  In SAS

  - LOGISTIC (maximum likelihood).
  - CATMOD (weighted least squares).

  For larger samples with categorical explanatory variables, the results are almost the same.

Example: High School and Beyond

$$X = \text{mean of 5 achievement test scores.}$$

$$\begin{aligned} Y &= \text{high school program type} \\ &= \begin{cases} 1 & \text{Academic} \\ 2 & \text{General} \\ 3 & \text{VoTech} \end{cases} \end{aligned}$$

So the logit model is

Academic vs (Gen & VoTech): $\text{logit}(Y \leq 1) = \alpha_1 + \beta x$

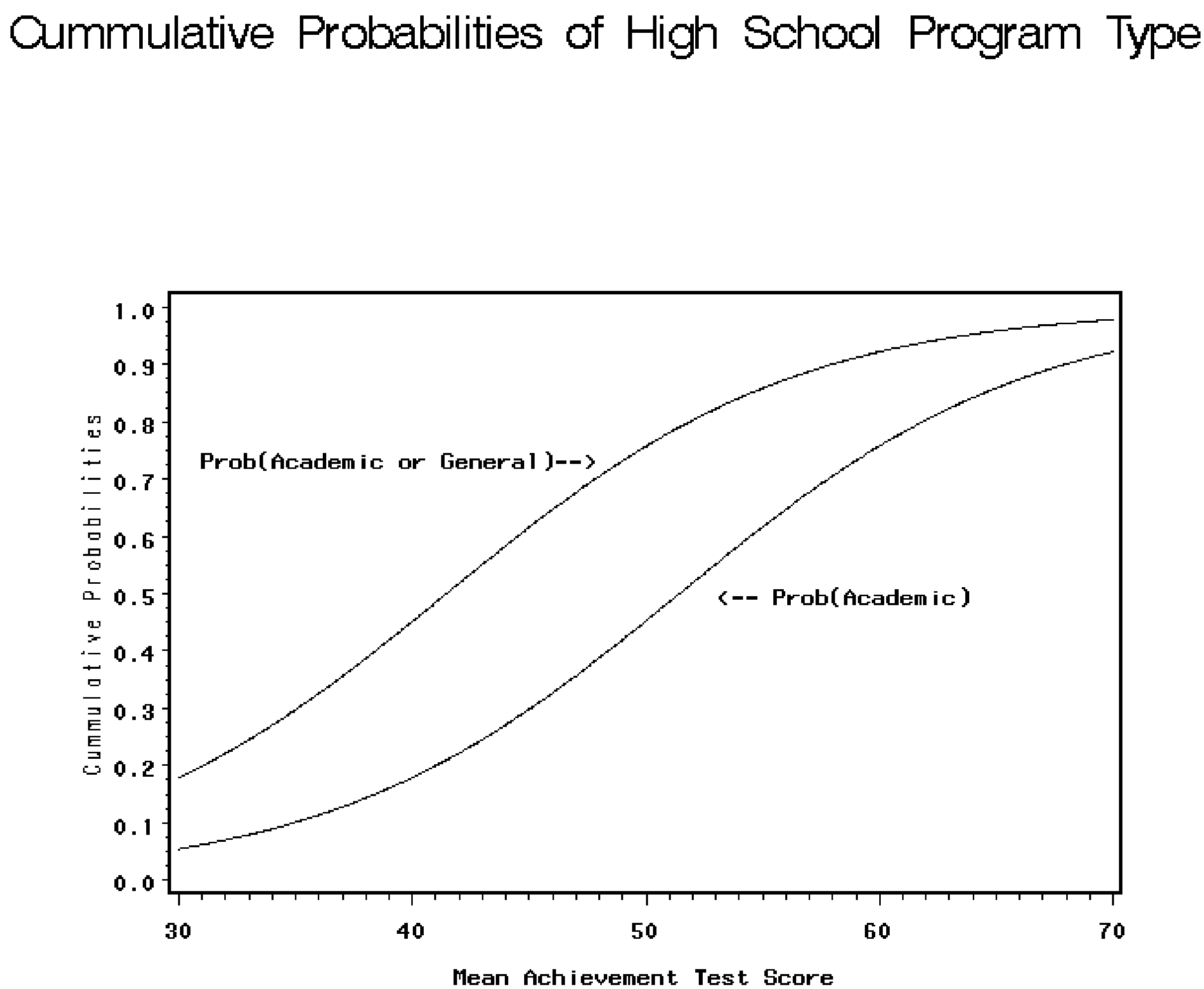
(Academic & Gen) vs VoTech: $\text{logit}(Y \leq 2) = \alpha_2 + \beta x$

Parameter	Estimate	e^{β}	ASE	Wald	p
α_1	-6.8408		.6118	125.04	$< .001$
α_2	-5.5138		.5866	88.37	$< .001$
β	.1330	1.142	.0118	127.64	$< .001$

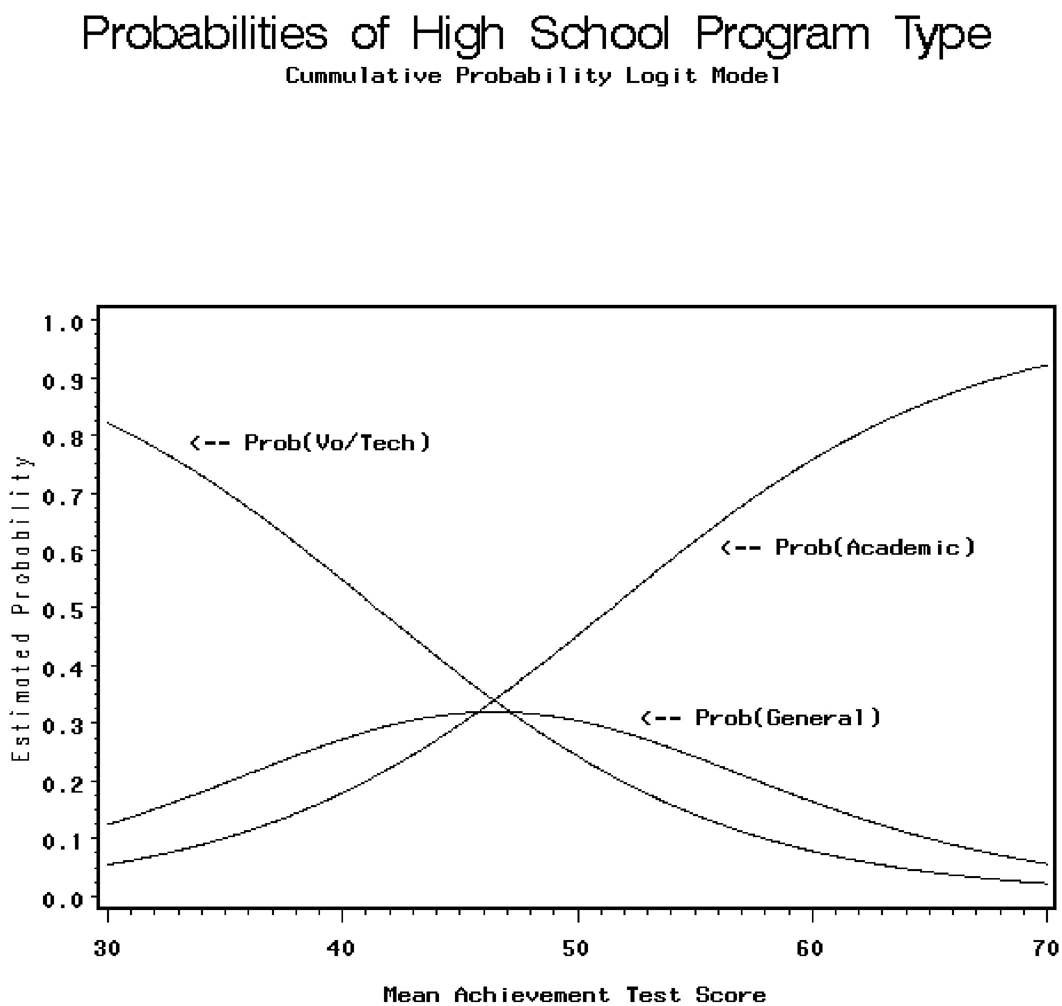
For a 10 point increase in mean achievement, the odds ratio (for either case) equals

$$\exp(10(.1330)) = 3.78$$

Plot of the estimated cumulative probabilities. . .



Plot of estimated probabilities of response category.



In the example, we would get the exact same results regarding interpretation if we had used

$$
\begin{aligned}
Y &= \text{high school program type} \\
&= \begin{cases} 1 & \text{VoTech} \\ 2 & \text{General} \\ 3 & \text{Academic} \end{cases}
\end{aligned}
$$

This reversal of the ordering of Y would

- Change the signs of the estimated parameters.
- Yield curves of cumulative probabilities that decrease (rather than increase).

SAS code for the example presented:

```
PROC LOGISTIC;
 MODEL hsp = achieve;
```

Fitting the cumulative logit model is the default if the response variables has more than 2 categories.

Final Comments on Cumulative Logit Models

Nice things about proportional odds model:

- It takes into account the ordering of the categories of the response variable.
- P(Y=1) is monotonically increasing as a function of x. (see figure of estimated probabilities).
- P(Y=J) is monotonically decreasing as a function of x. (see figure of estimated probabilities).
- Curves of probabilities for intermediate categories are uni-modal with the modes (maximum) corresponding to the order of the categories.
- The conclusions regarding the relationship between Y and x are not affected by the response category.

  The specific combination of categories examined does not lead to substantially difference conclusions regarding the relationship between responses and x.

If the proportional odds model does not fit well, then you can use the baseline (nominal) model and use the ordering of the responses in your interpretation of the model. For other possibilities, see Long (1997).

IRT connection: Samejima's (1969) graded response model for polytomous items is the same as the proportional odds model except that x is a latent continuous variable.

Adjacent–Categories Logit Models

for ordinal response

Rather than using all categories in forming logits, we can just use $J-1$ pairs of them.

To incorporate the ordering of the response, we use adjacent categories:

$$\log\left(\frac{\pi_{j+1}}{\pi_j}\right) \qquad j = 1, \ldots, J-1$$

The logit model for one (continuous) explanatory variable x is

$$\log\left(\frac{\pi_{j+1}}{\pi_j}\right) = \alpha_j + \beta_j x \qquad j = 1, \ldots, J-1$$

This is similar to the baseline category model in that

- Both α and β depend on the logit.
- When the explanatory variable is categorical, the logit model has an equivalent loglinear model.

Example: Data from Fienberg (1985) where the response variable is the education level of the rejectees from miltiary service.

Explanatory variables were Race (white, black), Age ($< 22, \geq 22$), and Father's education (grammar, some HS, HS graduate, not available).

A good fitting loglinear model was (RAF,EFR,EA), which had $df = 14$, $G^2 = 18.1$.

Estimated parameters for the adjacent category logit model (columns 1 and 3) and for the baseline category model (columns 2 and 3).

			Adjacent Category Logits		
				Baseline Logits	
			Grammar vs Some HS	Grammar vs HS grad	Some HS vs HS grad
$\hat{\alpha}$			-.17	.73	.90
$\hat{\beta}_i^R$	white		.86	.98	.12
	black		.00	.00	.00
$\hat{\beta}_j^A$	< 22		-.24	.20	.44
	≥ 22		.00	.00	.00
$\hat{\beta}_k^F$	1		-.08	-.71	-.63
	2		-.64	-1.10	-.47
	3		-.87	-1.25	-.39
	4		.00	.00	.00
$\hat{\beta}_{ik}^{RF}$	white	1	.06	.49	.43
		2	-.33	-.23	.09
		3	-.01	-1.06	-1.05
		4	.00	.00	.00
	black	1	.00	.00	.00
		2	.00	.00	.00
		3	.00	.00	.00
		4	.00	.00	.00

F = Father's education = 1 grammar, 2 some HS, 3 HS graduate, 4 not available.

A simpler logit model for adjacent categories:

$$\log\left(\frac{\pi_{j+1}}{\pi_j}\right) = \alpha_j + \beta x \qquad j = 1, \ldots, J-1$$

This is similar to the cummulative logit model in that the effect of x on Y is constant across logit (in this case, pairs of categories).

For two categories (say Y=1 and Y=4), the effect of x equals

$$\beta(4-1)$$

If you have just 1 categorical variable,

- The more complex adjacent categories logit model with β_j and the corresponding loginear model are saturated.
- The simpler model with β constant is not saturated.

The simpler model is equvialent to a loglinear with linear by linear term for the relationship between the explanatory and response variables.

Example of uniform association model.

General Social Survey (1994) data from before.

Item 1: A working mother can establish just as warm and secure of a relationship with her children as a mother who does not work.

Item 2: Working women should have paid maternity leave.

	Item 2				
	strongly agree	agree	neither	disagree	strongly disagree
Item 1	1	2	3	4	5
1	97	96	22	17	2
2	102	199	48	38	5
3	42	102	25	36	7
4	9	18	7	10	2

When using $u_i = i$ and $v_j = j$ as scores and fitting the independence loglinear model and the uniform assocation model

$$\log(\mu_{ij}) = \lambda + \lambda_i^I + \lambda_j^{II} + \beta^* ij$$

we got

Model/Test	*df*	G^2	p	Estimates
Independence	12	44.96	$< .001$	
Uniform Assoc	11	8.67	.65	$\hat{\beta}^* = .24$, $ASE = .0412$

Now suppose that we consider item 2 as the response variable and model adjacent category logits with the restriction that $\beta_j = \beta =$ a constant.

$$\begin{aligned}
\log\left(\frac{\mu_{i,j+1}}{\mu_{i,j}}\right) &= \lambda + \lambda_i^I + \lambda_{j+1}^{II} + \beta^* i(j+1) \\
&\quad -(\lambda + \lambda_i^I + \lambda_j^{II} + \beta^* ij) \\
&= (\lambda_{j+1}^{II} - \lambda_j^{II}) + \beta^*(ij + i - ij) \\
&= \alpha_j^* + \beta i
\end{aligned}$$

So the estimated local odds ratio equals (and the effect of response on item 1 on item 2 for adjacent categories)

$$e^{\hat{\beta}} = e^{.24} = 1.28$$

Continuation–Ratio Logits

for ordinal responses

In this approach, the order of the categories of the response variable is incorporated by forming a series of $(J-1)$ logits

$$\log\left(\frac{\pi_1}{\pi_2}\right), \log\left(\frac{\pi_1+\pi_2}{\pi_3}\right), \ldots, \log\left(\frac{\pi_1+\ldots+\pi_{J-1}}{\pi_J}\right)$$

or

$$\log\left(\frac{\pi_1}{\pi_2+\ldots+\pi_J}\right), \log\left(\frac{\pi_2}{\pi_3+\ldots+\pi_J}\right), \ldots, \log\left(\frac{\pi_{J-1}}{\pi_J}\right)$$

These are called "continuation–ratio logits".

When the models have different parameters for each logit, e.g.,

$$\alpha_j + \beta_j x$$

- Just apply regular binary logistic regression to each one.
- The fitting can be separate.
- The sum of the separate df and G^2 provide an overall global goodness of fit test and measure (same as simultaneous fitting).

Example from Fienberg (1985): Military rejects.

Reponse variable is **E** for respondent's education and the logits used are

$$\frac{\text{some HS}}{\text{grammar}}, \qquad \text{and} \qquad \frac{\text{HS grad}}{\text{some HS} + \text{grammar}}$$

Explanatory variables are
F for Father's education
A for respondent's age
R for respondent's race.

Logit models fit to each continuation ratio:

Loglinear	$\log(\frac{\mu_{ijk1}}{\mu_{ijk2}})$		$\log(\frac{\mu_{ijk3}}{\mu_{ijk1}+\mu_{ijk2}})$		Combined Fit	
Model	*df*	G^2	*df*	G^2	*df*	G^2
(FAR,E)	15	131.5	15	123.3	30	254.8
(FAR,EF)	12	97.9	12	64.7	24	162.6
(FAR,EA)	14	123.3	14	119.4	28	242.7
(FAR,ER)	14	49.0	14	103.8	28	152.8
(FAR,EF,EA)	11	91.9	11	60.3	22	152.2
(FAR,EF,ER)	**11**	**16.1**	11	35.6	22	51.7
(FAR,EA,ER)	13	43.7	13	98.7	26	142.4
(FAR,EF,EA,ER)	**10**	**12.4**	10	29.8	20	42.2
(FAR,EFA,ER)	7	9.3	7	23.2	14	32.5
(FAR,EFR,EA)	**7**	**11.5**	**7**	**7.0**	**14**	**18.5**
(FAR,ERA,EF)	9	8.6	9	29.7	18	38.3
(FAR,EFA,EFR)	4	8.5	4	1.2	8	9.7

Notes:

- The best fitting loglinear model, which we could use for baseline or adjacent categories model is

  (FAR,EFR,EA)

  which has $df = 14$, $G^2 = 18.5$, $p = .18$ for the combined fit, and for each separate logits
  $df = 7$, $G^2 = 11.5$, $p = .12$ and $df = 7$, $G^2 = 7.0$, $p = .43$.

- We can find simpler models that fit for the logit comparing Some HS and Grammar School:

  (FAR,EF,EA,ER) with $df = 10$, $G^2 = 12.4$, $p = .23$
  (FAR,EF,ER) with $df = 11$, $G^2 = 16.1$, $p = .14$.

  The likelihood ratio statistic for $H_o : \lambda_{jl}^{EA} = 0$ equals $G^2 = 16.1 - 12.4 = 3.7$ with $df = 1$, has $p = .05$.

The simpler of these two models states that given that the respondent did not complete high school, the odds of their completing some high school depends only on race and on their father's education.

The odds that a respondent completes high school requires a more complex model.