Statistical Inferences for Ordinal Variables in 2–way Tables
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Outline

Inference for ordinal variables.

- Linear trend instead of independence.
- Greater power with ordinal test.
- Choosing scores for categories.
- Trend tests for $2 \times J$ and $I \times 2$ tables

Testing Linear Trend instead of Independence

Consider the example from the GSS where we had 2 items both with ordinal response options:

- Item 1: A working mother can establish just as warm and secure a relationship with her children as a mother who does not work.
- Item 2: Working women should have paid maternity leave.

	Item2					
Item 2	Strongly Agree	Agree	Neither	Disagree	Strongly Disagree	
Strongly Agree	97	96	22	17	2	234
Agree	102	199	48	38	5	392
Disagree	42	102	25	36	7	212
Strongly Disagree	9	18	7	10	2	46
	250	415	102	101	16	884

GSS Example

Statistic		df	Value	p-value
Pearson Chi-square	X^2	12	47.576	$< .001$
Likelihood Ratio Chi-square	G^2	12	44.961	$< .001$

There is a "**linear trend**" in these data, so we may be able to describe this relationship using a single statistic:

(Pearson Product Moment) **Correlation**

$$r = \frac{\text{cov}(X, Y)}{s_X s_Y}$$

To compute r, we need **scores** for both the row (item 1) categories and the column (item 2) categories.

Category Scores and r

- For the categories of the row variable X:

$$u_1 \leq u_2 \leq \ldots \leq u_I$$

- For the categories of the column variable Y:

$$v_1 \leq v_2 \leq \ldots \leq v_J$$

When the scores have the same order as the categories, they are “monotone”.

Assume for now that we have scores. (we’ll discuss possible choices and their effect later).

Given scores $\{u_i\}$ and $\{v_j\}$, the correlation equals…

The Correlation for an $(I \times J)$ Table

$$r = \frac{\text{cov}(X,Y)}{s_x s_Y} = \frac{\sum_i \sum_j (u_i - \bar{u})(v_j - \bar{v}) n_{ij}}{\sqrt{\left[\sum_i \sum_j (u_i - \bar{u})^2 n_{ij}\right]\left[\sum_i \sum_j (v_j - \bar{v})^2 n_{ij}\right]}}$$

where

- Row mean

$$\bar{u} = \sum_i \sum_j u_i n_{ij}/n = \sum_i u_i n_{i+}/n$$

- Column mean

$$\bar{v} = \sum_i \sum_j v_j n_{ij}/n = \sum_j v_j n_{+j}/n$$

Properties of r for Contingency Table Data

- $-1 \leq r \leq 1$
- $r = 0$ corresponds to no (linear) relationship.
- The further r is from 0, the greater the strength of the relationship.
- Perfect association implies that $r = \pm 1$.
- $r = 1$ if all observations fall into cells on the "diagonal" that runs from the top left to bottom right of the table.
- $r = -1$ if all observations fall into cells on the "diagonal" that runs from the top right to bottom left of the table.

Testing Null Hypothesis of Independence

(i.e., no linear trend or $H_o : \rho = 0$)

Test statistic

$$M^2 = (n-1)r^2$$

- “Mantel–Haenszel” or “Cochran–Mantel–Haenszel” statistic.
- As n increase, M^2 gets larger.
- As r^2 increases, M^2 gets larger.
- Under independence, $\rho = 0$, $M^2 = 0$.
- For perfect association, $M^2 = (n-1)$.
- Larger values of M^2 provide more evidence against H_O.
- If H_O of independence is true, then M^2 is approximately chi-square distributed with $df = 1$.
- $\sqrt{M^2} = \sqrt{(n-1)}r$ is approximately distributed at $\mathcal{N}(0,1)$, which can be used to test one-sided alternative hypotheses that the correlation is > 0 or < 0.

Example: Testing $H_o : \rho = 0$

Try integer (Likert) scores for our categories:

Rows	Response	Columns
$u_1 = 1$	Strongly Agree	$v_1 = 1$
$u_2 = 2$	Agree	$v_2 = 2$
	Neither	$v_3 = 3$
$u_3 = 3$	Disagree	$v_4 = 4$
$u_4 = 4$	Strongly Disagree	$v_5 = 5$

$r = .203$ and $M^2 = (884 - 1)(.203)^2 = 36.26$

With $df = 1$, p–value for observed M^2 is $< .001$.

SAS INPUT to Compute M^2

- You must have two numeric variables, one for the rows ("row") and one for the columns ("col"), whose values are the scores.

```
DATA gss;
INPUT item1 $ item2 $ row col count;
DATALINES;
  strongagree   strongagree   1  1  97
  strongagree   agree         1  2  96
  :             :
  strongdis     strongdis     4  5   2
```

- For the TABLES command, use the numeric variables that contain the row and column scores.

```
PROC FREQ;
TABLES row*col / chisq measures;
```

- On the output:
  - "Mantel-Haenszel Chi-Square" is M^2.
  - "Pearson correlation" is r.

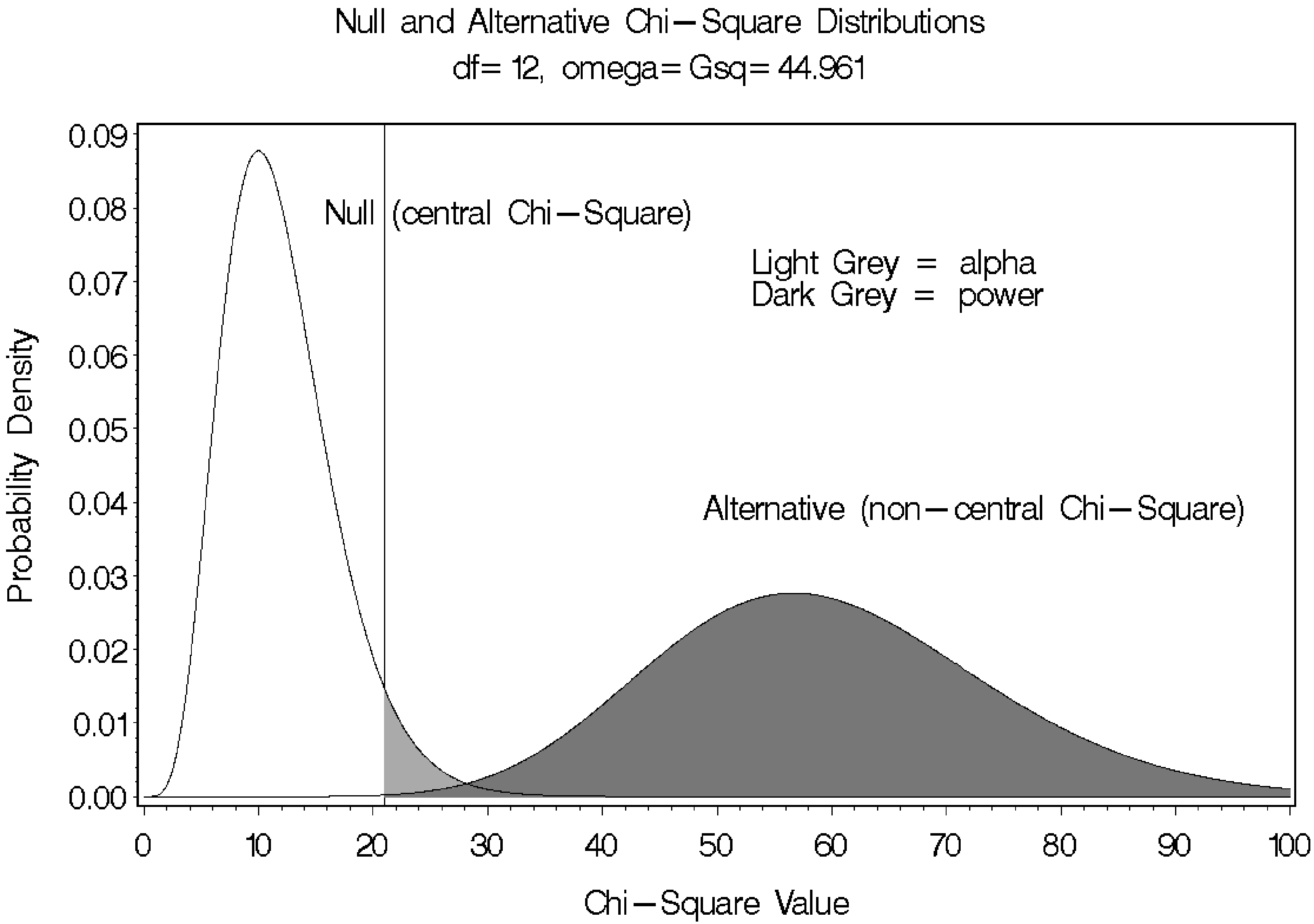
Extra Power with Ordinal Test

Statistic		df	Value	p-value
Pearson Chi-square	X^2	12	47.576	$< .001$
Likelihood Ratio Chi-square	G^2	12	44.961	$< .001$
Mantel-Haenszel Chi-square	M^2	1	36.261	$< .001$

- X^2 and G^2 are designed to detect any type association.
- M^2 is designed to detect a specific type of association.
- With ordinal data, we can summarize the association in terms of 1 parameter (i.e., r) rather than $(I-1)(J-1)$ of them (i.e., a set of $(I-1)(J-1)$ odds ratios).
- Advantages of M^2 over X^2 and G^2 when there is a positive or negative association between variables;
  - M^2 is more powerful.
  - M^2 tends to be about the same size as G^2 and X^2, but only has $df = 1$ rather than $df = (I-1)(J-1)$.
  - For small to moderate sample sizes, the true sampling distribution of the test statistics are better approximated for those with smaller df.

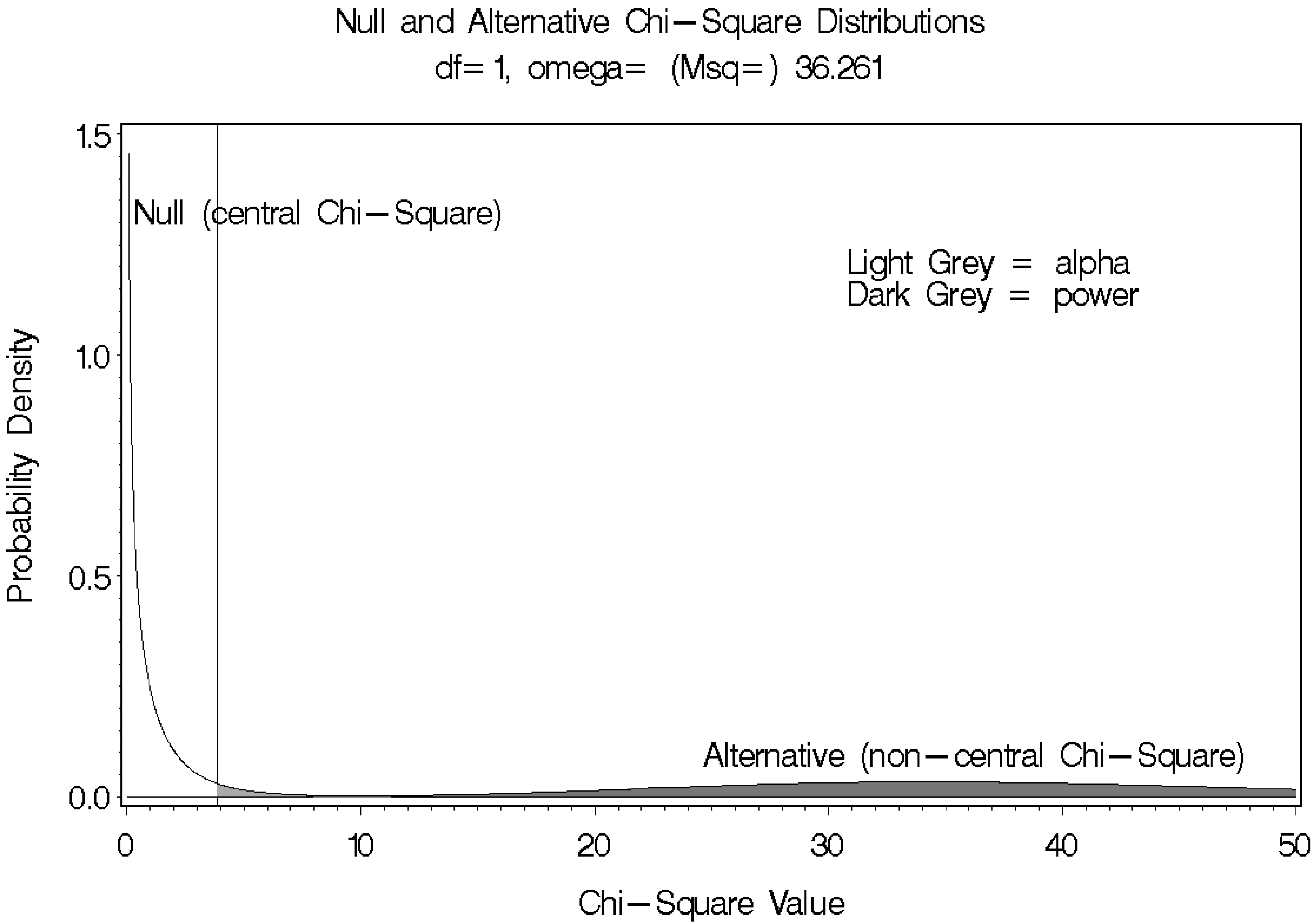
Power for Chi-square Tests: G^2

GSS data: For $G^2 = 44.961$, $df = 12 \longrightarrow$ power $= .99907$.



Power for M^2

For $M^2 = 36.261$, $df = 1 \longrightarrow$ power $= .99998$.



Computing Power

- π_{ij} = probabilities under the alternative model (which we'll take as the "saturated" model).
- π_{ij}^* = probabilities under the null hypothesis.
- N = total sample size.
- Note: $\mu_{ij} (= n_{ij}) = N\pi_{ij}$ and $m_{ij} = N\pi_{ij}^*$.
- "omega" (non-centrality parameter) for G^2

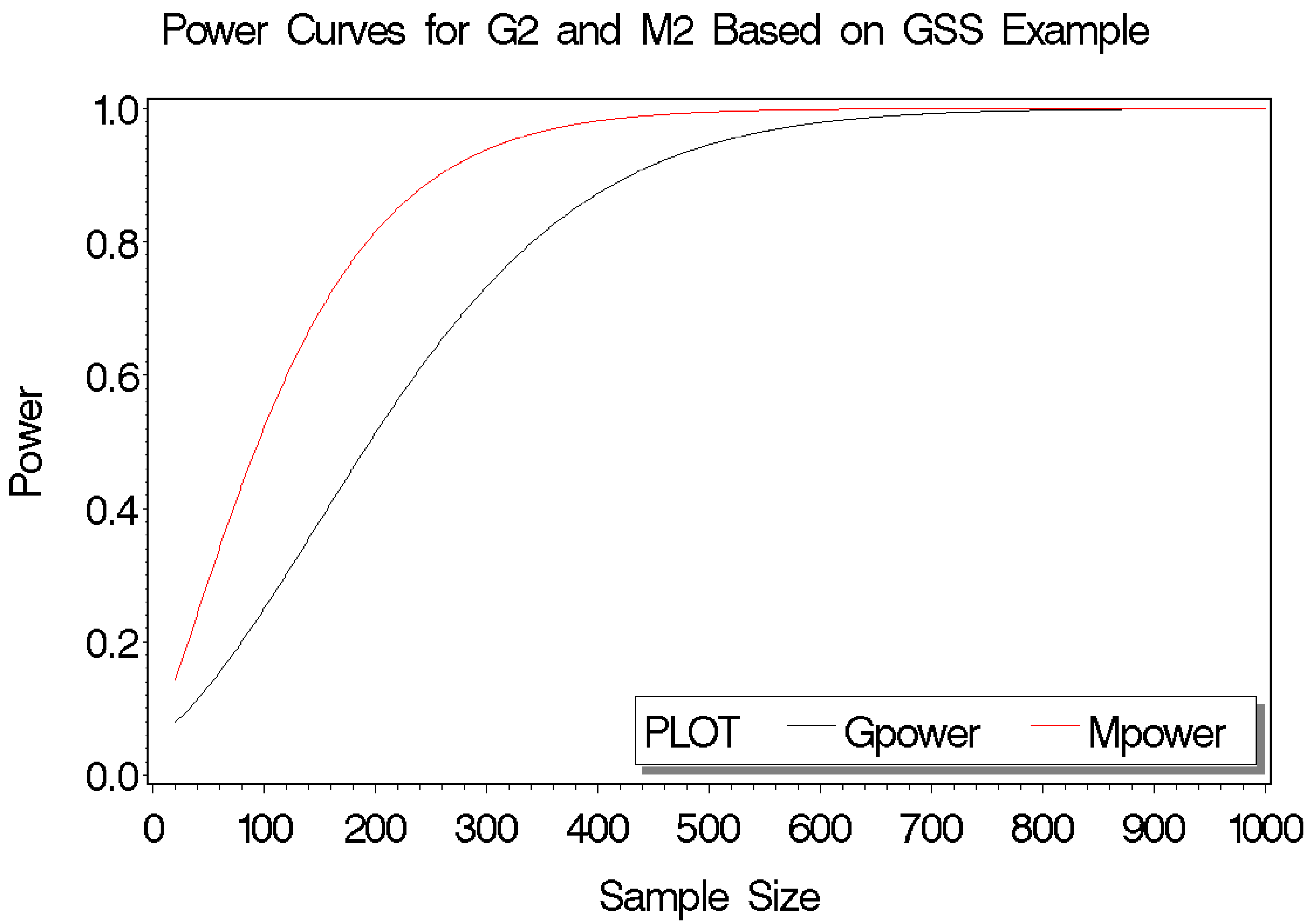
$$G^2 = 2N \sum_i \sum_j \pi_{ij} \log \frac{\pi_{ij}}{\pi_{ij}^*} = \omega$$

- "omega" for M^2

$$M^2 = (N-1)r^2 = \omega$$

- Sample Size and Power: $\uparrow N \Longrightarrow \uparrow \omega \Longrightarrow \uparrow$ Power

Power and Sample Size



Choice of Scores

- The choice of scores often does not make much difference with respect to the value of r and thus test results.
- For the GSS example, an alternative scoring that changed the relative spacing between the scores leads to an increase of r from $.203$ (from equal spacing) to $.207$ (from one possible choice for unequal spacing).
- The "best" scores for GSS table that lead to the largest possible correlation, yields $r = .210$. (Score from correspondence analysis).
- Different scoring tends to have a larger difference when the margins of the tables are unbalanced; that is, when there are some vary large margins and some relatively small ones.

Choice of Scores: Example 2

- Data from Farmer, Rotella, Anderson & Wardrop (1996) on gender differences in science careers. The data consist of a cross-classification of individuals by their gender and the prestige level of their occupation. (All subjects/individuals in this study had chosen a career in a science related field).

	Prestige Level of Occupation						
Gender	40–49	50–59	60–69	70–79	80–89	90–99	
Women	22	2	12	11	10	4	61
Men	3	0	11	6	25	7	52
	25	2	23	17	35	11	113

-
```
Statistic                     DF      Value          Prob
-----------------------------------------------------------
Chi-Square                     5     24.640         0.001
Likelihood Ratio Chi-Square    5     27.372         0.001
Mantel-Haenszel Chi-Square     1     19.840         0.001
Pearson Correlation                    .421
```

Different Possible Choices of Scores

- Equal Spacing. This is the SAS default.
- Midranks are a "no thought" approach to selecting scores.
  - Rank all observations on each variable and then use the ranks to compute the correlation — "Spearman's Rho" or the rank order correlation.
  - All individuals in the same category get the same rank, which equals the "midrank" for them.
  - e.g., Farmer et al data:

Category	Midrank/Score
40–49	$(1+25)/2 = 13.0$
50–59	$(26+27)/2 = 26.5$
60–69	$(28+50)/2 = 39.0$
70–79	$(51+67)/2 = 59.0$
80–89	$(68+102)/2 = 85.0$
90–99	$(103+113)/2 = 108.0$

  - In SAS to mid-ranks: PROC FREQ;
    TABLES row*col / cmh1 scores=ridits;

Different Possible Choices of Scores

- Midranks (continued)
  - In our example, different scores don't change our conclusion, if margins are really extreme (see example in Agresti), it can change results.
- Midpoints. When a categorical variable is a discretized numerical one, a good choice of scores often the midpoint.

  In our example, this leads to equal spacing.
- Use what you know about the data and your best guess as to what the relative spacing should be between the categories.
- Analytical method. Use row-column or "RC" association model or correspondence analysis.
- Try a few different ones to see if it makes a difference — a "sensitivity analysis".
- My preference: model the association.

Example and Results with Different Scores

Summary of Results using different methods

Scoring	M^2	p	Pearson r	ASE
Midranks (Ridits)	19.142	$< .01$	$.413$	0.081
Equally spaced	19.840	$< .01$	$.421$	0.077
Unequal spacing*	18.281	$< .01$	$.404$	0.078
Unequal spacing†	21.664	$< .01$	$.440$	$.076$

$*$ Column scores were -4, -2, -1, 1, 2, and 4
$\dagger$ Column scores were -4, -3, -0.5, 0.5, 3, 4

Didn't really make much of a difference. . . now for one where scores do matter.

School of Psychiatric Thought

Wrong ordering of scores:

```
        SCHOOL     ORIGIN
                     1        2        3
     Frequency|bio      |env      |comb     |
     ---------+--------+--------+--------+
1    eclectic |     90 |     12 |     78 |
     ---------+--------+--------+--------+
2    medical  |     13 |      1 |      6 |
     ---------+--------+--------+--------+
3    psychan  |     19 |     13 |     50 |
     ---------+--------+--------+--------+

Statistic                       DF     Value        Prob
-------------------------------------------------------
Chi-Square                       4    22.378       0.001
Likelihood Ratio Chi-Square      4    23.036       0.001
Mantel-Haenszel Chi-Square       1    10.736       0.001

Pearson Correlation                    0.195  (ASE=0.056)
```

A Better Ordering of Categories

Uniform Scores for row and column with good ordering:

	Frequency	bio 1	env 3	comb 2	Total
eclectic	2	90	12	78	180
medical	1	13	1	6	20
psychan	3	19	13	50	82

Statistic	DF	Value	Prob
Chi-Square	4	22.378	0.001
Likelihood Ratio Chi-Square	4	23.036	0.001
Mantel-Haenszel Chi-Square	1	20.260	0.001

Pearson Correlation 0.269 (ASE=0.056)

A Better Ordering and Scores: RC Model

Scale values from RC association model (scores are estimated from the data):

Statistic	DF	Value	Prob
Chi-Square	4	22.378	0.001
Likelihood Ratio Chi-Square	4	23.036	0.001
Mantel-Haenszel Chi-Square	1	22.042	0.001

Statistic	Value	ASE
Pearson Correlation	0.280	0.055

Trend Tests

Situation: the row variable X is an explanatory variable and the column variable Y is a response/outcome variable.

- When one variable just has two levels (e.g., Farmer et al), we can assign the categories any two distinct values, e.g., 0 and 1, -1 and 1, 0 and 5000 — the choice does not effect r.
- Binary X: (i.e, $u_1 = 0$ and $u_2 = 1$) and polytomous ordinal Y with scores $v_1, \ldots, v_J$.
- The term in the covariance $\sum_i \sum_j u_i v_j n_{ij}$ between X and Y simplifies to
$$\sum_i \sum_j u_i v_j n_{ij} = \sum_j v_j n_{2j}$$
- When this is divided by the number of individuals in the 2nd row, we get
$$\bar{v}(i=2) = \sum_j v_j n_{2j} / n_{2+}$$
- So, testing a linear trend in this case is the same as testing whether the mean on Y is the same or different for the two rows.

Trend Test for $2 \times J$ Tables

- Testing a linear trend in this case is the same as testing whether the mean on Y is the same or different for the two rows.
- When midranks are used, the test for linear trend using M^2 is the same as the *Wilcoxon* and *Mann-Whitney* non-parametric tests for mean differences.
- Now for the other case... $I \times 2$ Tables

Trend Test for $I \times 2$ Tables

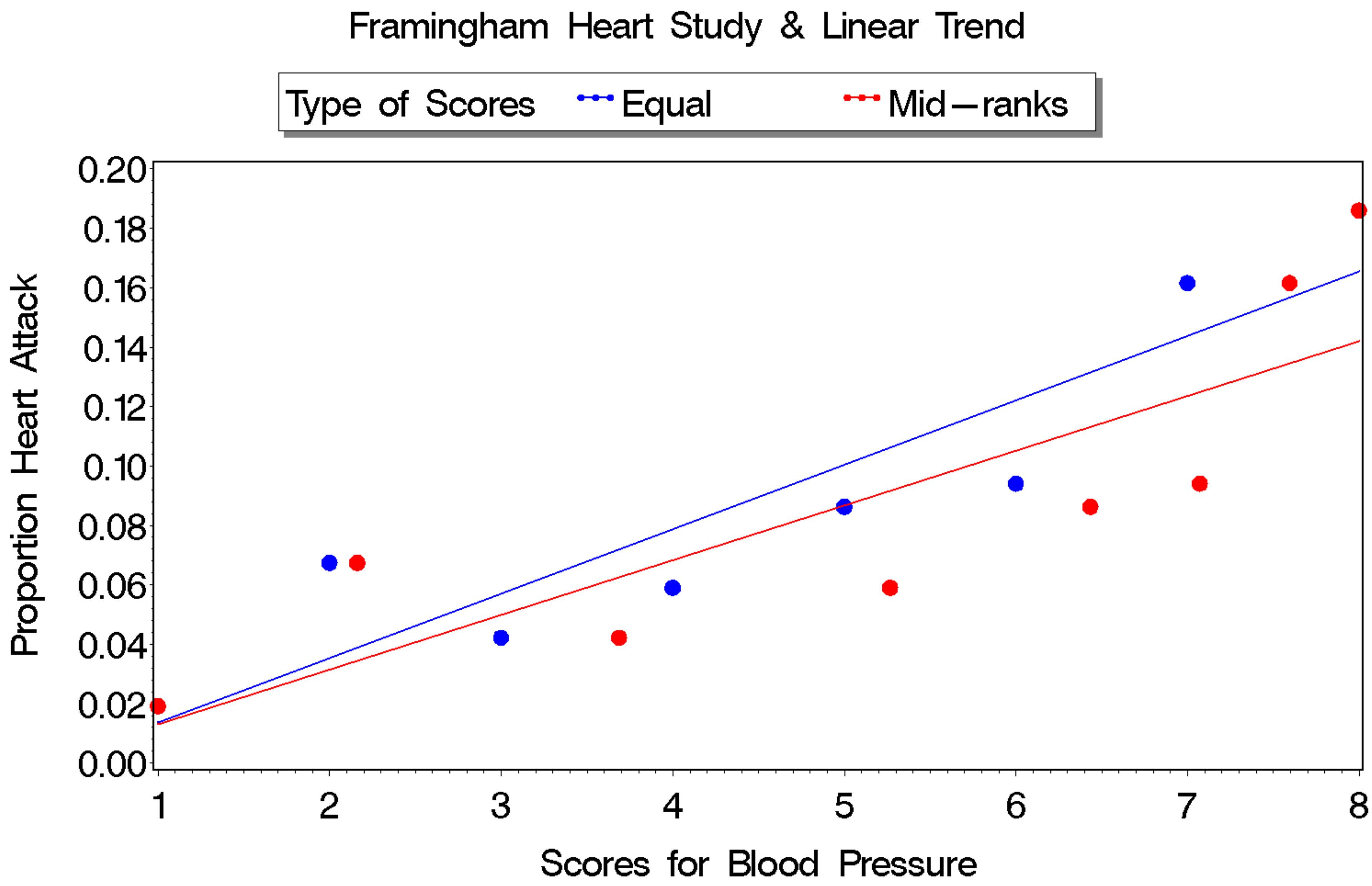
Situation: Polytomous ordinal X with scores $u_1, \ldots, u_I$ and binary Y ($v_1 = 0$ and $v_2 = 1$).

- This test detects whether the proportion classified as (for example) Y_1 increases (or decreases) linearly with X.
- Cochran–Armitage trend test is the $I \times 2$ version of M^2. You can specify choice of scores (SAS default: scores=table).
- Example: The Framingham heart study from Cornfield (1962). 40–59 year old males from Framingham, MA were classified on several factors. At a 6 year follow-up,

Blood pressure	Heart disease Present	(%)	Absent	Total
< 117	3	(.02)	153	156
117–126	17	(.07)	235	252
127–136	12	(.04)	272	284
137–146	16	(.06)	255	271
147–156	12	(.09)	127	139
157–166	8	(.09)	77	85
167–186	16	(.16)	83	99
> 186	8	(.19)	35	43

- Is there is significant linear trend?

Look at the Data



Final Comments: Cochran–Armitage Trend Test

- Cochran–Armitage trend test is analogous to testing the slope in a linear (probability) regression model:

$$\pi_i = \alpha + \beta(\text{category score})_i + \epsilon_i$$

- Cochran–Armitage trend test is the "score test" for β.
- Let $z \sim \mathcal{N}(0, 1)$,

$$\chi^2(\text{independence}) = z^2 + \chi^2(\text{lack of linear trend}).$$

The Cochran–Armitage trend test statistic equals z.